

$\log \left(\begin{array}{c} \text{TESTING} \\ \text{PROGRAM} \end{array} \right)$

favonia

Testing map

$\text{map} : \forall \alpha. \forall \beta. (\alpha \rightarrow \beta) \rightarrow \text{list}(\alpha) \rightarrow \text{list}(\beta)$

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$$\alpha^* = \beta^* = \mathbb{N}$$

identity function and $[0, 1, \dots, n-1]$

Testing exists

exists : $\forall \alpha. (\alpha \rightarrow 2) \rightarrow \text{list}(\alpha) \rightarrow 2$

$$\alpha^* = 2$$

identity function and all lists

Testing pick

$\text{pick} : \forall \alpha. \alpha \times \alpha \rightarrow \alpha$

$$\alpha^* = 2$$

(true, false)

Theorem [Bernardy *et al.*]

$$p : \forall \alpha. (\underbrace{F(\alpha) \rightarrow \alpha}_{\text{POS}}) \times (\underbrace{G(\alpha) \rightarrow K}_{\text{POS}}) \rightarrow \underbrace{H(\alpha)}_{\text{POS}}$$


$$\alpha^* = \mu F \quad \textit{if it exists}$$

$$\rho^* : F(\alpha^*) \rightarrow \alpha^* \quad \textit{catamorphism}$$

“roll”

Theorem [Bernardy *et al.*]

$$p : \forall \alpha. \underbrace{(F(\alpha) \rightarrow \alpha)}_{\text{POS}} \times \underbrace{(G(\alpha) \rightarrow K)}_{\text{POS}} \rightarrow \underbrace{H(\alpha)}_{\text{POS}}$$

embed  *project*

$$p : \forall \alpha. \underbrace{P(\alpha)}_{\text{POS}} \rightarrow \underbrace{H(\alpha)}_{\text{POS}}$$

My Problems with Embedding/Projection

Type μF buried under layers
of existence theorems.

Error-prone: *e.g.*, wrongly
assuming $\text{list}(\alpha) \subseteq \mathbb{N} \times (\mathbb{N} \rightarrow \alpha)$.

$p : \forall \alpha. P(\alpha) \rightarrow H(\alpha)$

POS

Logarithmic Conjecture

$$p : \forall \alpha. P(\alpha) \rightarrow H(\alpha)$$

POS

$$\alpha^* = \mu \alpha. \log_{\alpha} P(\alpha) \quad \text{if it exists}$$

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$$\alpha^* = \mu \alpha. \log_{\alpha} P(\alpha) \quad \text{if it exists}$$

Goal: simple and direct calculation

Testing pick

pick : $\forall \alpha. \alpha \times \alpha \rightarrow \alpha$

$$\alpha^* = \log_{\alpha}(\alpha \times \alpha) = \log_{\alpha}(\alpha^2) = 2$$

“all locations of α ”

“all ways to generate an α -element”

$$\log_{\alpha} \alpha = 1$$

$$\log_{\alpha} K = 0$$

$$\log_{\alpha}\alpha = 1$$

$$\log_{\alpha}K = 0$$

$$\log_{\alpha}(A \times B) = \log_{\alpha}A + \log_{\alpha}B$$

$$\log_{\alpha}(A + B) = \log_{\alpha}A \cup \log_{\alpha}B$$

$$A \cup B \approx \text{“max}(A, B)\text{”}$$

can be $A+B$, ideally more optimized

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$$\log_{\alpha}(A \cup B) = \log_{\alpha}A \cup \log_{\alpha}B \quad \textit{optional}$$

$$\log_{\alpha} \alpha = 1 \quad \log_{\alpha} K = 0$$

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$$A \cup B \approx \text{“max}(A, B)\text{”}$$

can be $A+B$, ideally more optimized

$$\log_{\alpha} (A \cup B) = \log_{\alpha} A \cup \log_{\alpha} B \quad \textit{optional}$$

$$\log_{\alpha} (A \rightarrow B) = A \times \log_{\alpha} B$$

“ B^A ”

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$$P(\alpha) = (\alpha \rightarrow \beta) \times \text{list}(\alpha)$$

Testing map

$\text{map} : \forall \alpha. \forall \beta. (\alpha \rightarrow \beta) \rightarrow \text{list}(\alpha) \rightarrow \text{list}(\beta)$

$$P(\alpha) = (\alpha \rightarrow \beta) \times \text{list}(\alpha)$$

$$\log_{\alpha} P(\alpha) = \log_{\alpha}(\alpha \rightarrow \beta) + \log_{\alpha} \text{list}(\alpha)$$

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$$P(\alpha) = (\alpha \rightarrow \beta) \times \text{list}(\alpha)$$

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$$= \alpha \times \log_{\alpha} \beta$$

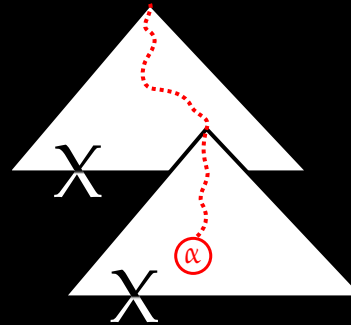
$$= \alpha \times 0$$

$$\cong 0$$

Rules of *Recursive* Logarithm

$$X \cong P(X)$$

$$\log_{\alpha} X \cong \log_{\alpha} P(X)$$

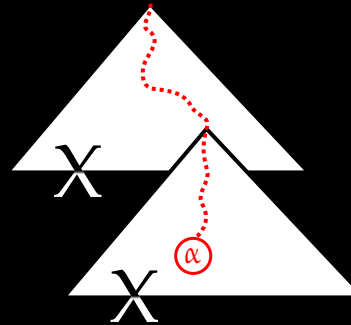


Rules of *Recursive* Logarithm

$$X \cong P(X)$$
$$\log_{\alpha} X \cong \log_{\alpha} P(X)$$



$$\log_{\alpha} (\mu X.P) = \mu X'.(\log_{\alpha} P)[\mu X.P/X]$$
$$\log_{\alpha} X = X'$$



Rules of *Recursive* Logarithm

$$\text{list}(\alpha) = \mu X. 1 + \alpha \times X$$

$$\begin{aligned}\log_{\alpha}(\text{list}(\alpha)) &= \mu X'. 0 \cup (1 + X') \\ &\approx \mu X'. 1 + X' \\ &= \mathbb{N}\end{aligned}$$

$$\log_{\alpha}(\mu X.P) = \mu X'. (\log_{\alpha} P) [\mu X.P / X]$$

$$\log_{\alpha} X = X'$$

I will assume $0 \cup A = A$ from now on

Testing map

$\text{map} : \forall \alpha. \forall \beta. (\alpha \rightarrow \beta) \rightarrow \text{list}(\alpha) \rightarrow \text{list}(\beta)$

$$P(\alpha) = (\alpha \rightarrow \beta) \times \text{list}(\alpha)$$

$$\log_{\alpha} P(\alpha) = \underbrace{\log_{\alpha}(\alpha \rightarrow \beta)}_{\cong 0} + \underbrace{\log_{\alpha} \text{list}(\alpha)}_{\approx \mathbb{N}}$$

Logarithmic Conjecture

$$p : \forall \alpha. P(\alpha) \rightarrow H(\alpha)$$

POS

$$\alpha^* = \mu \alpha. \log_{\alpha} P(\alpha) \quad \text{if it exists}$$

Goal: simple and direct calculation

Non-regular Recursion and Change of Bases / Chain Rules

$$N(\alpha) \cong 1 + \alpha \times N(\alpha \times \alpha)$$

$$\alpha - \alpha^2 - \alpha^4 - \alpha^8 - 1$$

Non-regular Recursion and Change of Bases / Chain Rules

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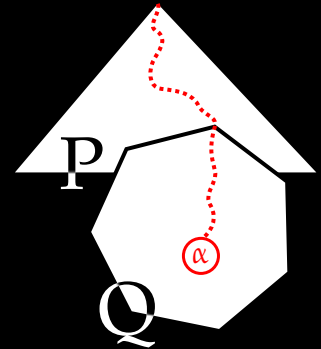
$$\alpha - \alpha^2 - \alpha^4 - \alpha^8 - 1$$

$$\log_{\alpha} N(\alpha) \cong 1 + \text{"log}_{\alpha \times \alpha} N(\alpha \times \alpha)" \times \log_{\alpha}(\alpha \times \alpha)$$

$$N'(\alpha) \cong 1 + N'(\alpha \times \alpha) \times 2$$

Non-regular Recursion and Change of Bases / Chain Rules

$$\log_{\alpha} P(Q(\alpha)) = P'(Q(\alpha)) \times Q'(\alpha)$$



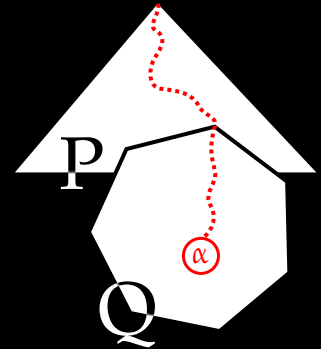
Non-regular Recursion and Change of Bases / Chain Rules

$$\log_{\alpha} P(Q(\alpha)) = P'(Q(\alpha)) \times Q'(\alpha)$$

*Should work for arbitrary P
with more equations*

$$\log_{\alpha} ((\alpha^2)^3) = 3 \times 2$$

$$\log_{\alpha} (\alpha^2 \times \alpha^2 \times \alpha^2) = 2 + 2 + 2$$



Non-regular Recursion and Change of Bases / Chain Rules

$$F(\alpha) \cong 1 + \alpha + \text{list}(\alpha) \times F(\alpha^2 + \alpha^3) \times \text{list}(\alpha)$$

(finger trees)

$\text{list}(\alpha)$ may be optimized to $\alpha + \alpha^2 + \alpha^3 + \alpha^4$

Non-regular Recursion and Change of Bases / Chain Rules

$$F(\alpha) \cong 1 + \alpha + \text{list}(\alpha) \times F(\alpha^2 + \alpha^3) \times \text{list}(\alpha)$$

(finger trees)

$$F'(\alpha) \cong 1 \cup (\mathbb{N} + F'(\alpha^2 + \alpha^3) \times (2 \cup 3) + \mathbb{N})$$

$\text{list}(\alpha)$ may be optimized to $\alpha + \alpha^2 + \alpha^3 + \alpha^4$

Corollary of Conjecture

$$p : \forall \alpha. (F(\alpha) \rightarrow \alpha) \times (G(\alpha) \rightarrow K) \rightarrow H(\alpha)$$

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$$\begin{aligned} & \log_{\alpha}((F(\alpha) \rightarrow \alpha) \times (G(\alpha) \rightarrow K)) \\ = & \underbrace{\log_{\alpha}(F(\alpha) \rightarrow \alpha)}_{\cong F(\alpha)} + \underbrace{\log_{\alpha}(G(\alpha) \rightarrow K)}_{\approx 0} \end{aligned}$$

$$\alpha^* \cong \mu\alpha. (F(\alpha) + 0) \cong \mu F$$

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How to plug in ρ^* , the catamorphism (“roll”)?

Upgraded Logarithmic Conjecture

$$p : \forall \alpha. P(\alpha) \rightarrow H(\alpha)$$

POS

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Upgraded Logarithmic Conjecture

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$$\alpha^* = \mu \alpha. P'(\alpha) \quad \text{if it exists}$$

$$\rho^* : P'(\alpha^*) \rightarrow \alpha^* \quad \text{catamorphism} \\ \text{“roll”}$$

wants: $\rho^+ : \text{something} \rightarrow P(\alpha^*)$

has: $\rho^* : P'(\alpha^*) \rightarrow \alpha^*$

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ρ^* can fill in
strictly positive positions

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exists : $\forall \alpha. (\alpha \rightarrow 2) \rightarrow \textit{list}(\alpha) \rightarrow 2$

has: $\rho^* : P'(\alpha^*) \rightarrow \alpha^*$

wants: $\rho^+ : \textit{something} \rightarrow P(\alpha^*)$

ρ^* can fill in
strictly positive positions

exists : $\forall \alpha. (\alpha \rightarrow 2) \rightarrow \textit{list}(\alpha) \rightarrow 2$

something: everything else

Killing the Strictly Positive

$$\begin{aligned}\alpha^- &= 1 & K^- &= K \\ (A \rightarrow B)^- &= A \rightarrow B^-\end{aligned}$$

$$\begin{aligned}(A \times B)^- &= A^- \times B^- & (A \cup B)^- &= A^- \cup B^- \\ (A + B)^- &= A^- + B^- & (\mu\beta.A)^- &= \mu\beta.A^-\end{aligned}$$

$$\rho^+ : \underset{\text{residual}}{\mathbf{P}^-(\alpha^*)} \rightarrow \underset{\text{full}}{\mathbf{P}(\alpha^*)}$$

Corollary of *Upgraded* Conjecture

$$p : \forall \alpha. (F(\alpha) \rightarrow \alpha) \times (G(\alpha) \rightarrow K) \rightarrow H(\alpha)$$

$$\begin{aligned} & ((F(\alpha) \rightarrow \alpha) \times (G(\alpha) \rightarrow K))^- \\ &= (F(\alpha) \rightarrow 1) \times (G(\alpha) \rightarrow K) \\ &\quad \underline{\cong 1} \end{aligned}$$

$$\rho^+(-, g) \approx (\rho^*, g)$$

with appropriate rules (omitted)

ρ^+ puts ρ^* at the right places

Problems: Suboptimal Types

$\text{exists} : \forall \alpha. (\alpha \rightarrow 2) \rightarrow \text{list}(\alpha) \rightarrow 2$

$\alpha^* = \mathbb{N}$, but $\alpha = 2$ with `id` seems better
similarly “all”, “toString”, “filter”, etc.

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$\text{length} : \forall \alpha. \text{list}(\alpha) \rightarrow \mathbb{N}$

$\alpha^* = \mathbb{N}$, but $\alpha = 1$ suffices

Problems: Suboptimal Types

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Current: full distinctiveness

Next: *inspectability*

Future Work

Future Work

Conjectures \rightarrow Theorems

Future Work

Conjectures \rightarrow Theorems

Greatest fixed points, etc.

Future Work

Conjectures → Theorems

Greatest fixed points, etc.

Unknown *datatypes*, not just functions

Checker for parts of CMU 15-210 in 2015
General theory under development

Some Related Work

Testing Polymorphic Properties

Jean-Philippe Bernardy, Patrik Jansson, and Koen Claessen

Categories of Containers

Michael Abbott, Thorsten Altenkirch, and Neil Ghani

A semantics for shape

C. Barry Jay

Species and Functors and Types, Oh My!

Brent A. Yorgey

*Unlike derivatives,
logarithms differentiate.*