towards efficient cubical type theory
Scientific Study of efficiency
into the cubes
\( i : \mathbb{I} \vdash M : A \)
i: \Pi, j: \Pi \vdash M: A
\[ i : \mathbb{I}, j : \mathbb{I}, k : \mathbb{I} \vdash M : A \]
i : I, j : I, k : I, l : I ⊢ M : A
$i_1 : \Pi, \ldots, i_n : \Pi \vdash M : A$
Kan filling/composition structure
coercion/transport
\[
\mathrm{coe}^{0\rightsquigarrow 1}[i.A](M) : A[1/i]
\]
$\text{ coerce}^0 \gamma [i.A](M) : A[j/i]$
\[ \text{coercion/transport} \]
homogeneous composition
homogeneous composition
homogeneous composition
homogeneous composition
hcom$^{0 \rightsquigarrow 1}$[A](M)

$[i=0 \leftrightarrow \ldots, i=1 \leftrightarrow \ldots, j=1 \leftrightarrow \ldots] : A$

*homogeneous composition*
$\text{hcom}^{0 \rightarrow i} [A](M)$

$[i=0 \leftrightarrow ..., i=1 \leftrightarrow ..., j=1 \leftrightarrow ...] : A$

*homogeneous composition*
hcom$^{0\rightarrow 1}$[A](M) 
$[i=0 \leftrightarrow \ldots, i=1 \leftrightarrow \ldots, i=j \leftrightarrow \ldots]$ : A

homogeneous composition
univalence and higher indexed inductive types with canonicity

[CCHM, AFH, ABCFHL, CHM, Cavallo & Harper]
see also Coquand's notes
extension types
\(<i>P : Path[i.A](M,N)\)
\[
<i>P : [i] A \ [i=0 \Leftrightarrow M, \ i=1 \Leftrightarrow N]
\]

extension types

[Shulman & Riehl]
(P : [i] A []) \rightarrow 
(Q : [i] A [i=0 \leftrightarrow P 1]) \rightarrow 
[i] A [i=0 \leftrightarrow P 0, i=1 \leftrightarrow Q 1]
\[ [i \ j] \ A \ [i=0 \leftrightarrow \ldots, \ i=j \leftrightarrow \ldots] \]
\[ \text{coe}[i.[j]A[]](<j>\succ M) = <j> \text{coe}[i.A](M) \]

fewer fixers, fewer fixes
empty systems
$hcom[A](M)[]$
hcom[A](M)[]

= M with *regularity*

easy to have regularity without univalent Kan universes & HITs

see summary in [Swan] 1808.00920
why do we have empty systems?

- the lack of coe (in some variants)
- “∀” operator (in some variants)
\[ \text{com}[i.A] \]

\[ \text{coe}[i.A] + \text{hcom}[A] \]
com[i.A](M)[]
coe[i.A] + hcom[A]
coercion without coe
separating coe and hcom

- makes HITs possible and
- kills a major source of empty systems
kill empty systems *completely*?

restrict shapes of hcom to cofibrations that are, equivalently,

- [geometry] covering every point; or
- [syntax] *true* under all closed substitutions; or
- [topos] \( \{ \varphi \in \text{Cof} \mid \neg \neg \llbracket \varphi \rrbracket \} \)

thanks to Christian Sattler for the topos formulation
- variants based on cartesian cubes: CHTT [AFH,CH], RedPRL, redtt, ...
- variants based on de morgan cubes: maybe? ask Andrea Vezzosi

difficulty: still need to handle arbitrary cofibrations (due to “∀”)
open: generality? is the extra complexity worth it?
kind

semilattices
Kan types

pretypes
discrete types

Kan types

pretypes

constant presheaves
discrete types

the *entire* “ETT”, including equality types, can be embedded while coexisting with other cubical features
discrete types

Kan types

pretypes

types with trivial hcom

types with coe

type families with trivial coe

type families with coe

more can be added; ask Evan Cavallo about trivial coe/hcom
kinds

*automatic* association of structure or properties with (families of) types (*cf.* the [LOPS] style)

needs a meet semilattice; better if it is Heyting
if $A : U_{k_1}, A : U_{k_2}, \ldots, A : U_{k_n}$, then $A : U_{k^*}$?

what's missing from $A : U_k$ to reach $A : U_{k^*}$?

meet$_i(k_i) \leq k^*$

$k \rightarrow k^*$
data pushout where

| inl (a : A) |
| inr (b : B) |
| push (i : \(I\)) (c : C) [i=0 \(\Rightarrow\) inl (f c), i=1 \(\Rightarrow\) inr (g c)]
$$\text{coe}(\text{inl}(a)) = \text{inl}(\text{coe}(a))$$
$$\text{coe}(\text{inr}(b)) = \text{inr}(\text{coe}(b))$$
$$\text{coe}(\text{inl}(a)) = \text{inl}(\text{coe}(a))$$

$$\text{coe}(\text{inr}(b)) = \text{inr}(\text{coe}(b))$$

$$\text{coe}(\text{push}_i(c)) \neq \text{push}_i(\text{coe}(c))$$
\[\text{coe}(\text{inl}(a)) = \text{inl}(\text{coe}(a))\]
\[\text{coe}(\text{inr}(b)) = \text{inr}(\text{coe}(b))\]
\[\text{coe}(\text{push}_i(c)) \neq \text{push}_i(\text{coe}(c))\]
\[ \text{coe}(\text{inl}(a)) = \text{inl}(\text{coe}(a)) \]
\[ \text{coe}(\text{inr}(b)) = \text{inr}(\text{coe}(b)) \]
\[ \text{coe}(\text{push}_i(c)) = \text{hcom}...\text{(omitted)} \]

naive coercion is fine when \( f \) and \( g \) are "clean" (ex: joins) or when \( A \) and \( B \) are \textit{discrete} (ex: suspensions)

ask Evan Cavallo about cleanliness
what's next?

- make great proof assistants
- optimize Kan operations of universes
- recover regularity as much as possible
- finish all the meta-theorems