Colimits in Homotopy Type Theory

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Abstract

We develop the theory of (homotopy) colimits inside homotopy type theory. The heart of our work characterizes the connection between colimits in *coslices* of a universe, called *coslice colimits*, and colimits in the universe (i.e., ordinary colimits). To derive this characterization, we find an explicit construction of colimits in coslices that is tailored to reveal the connection. We use the construction to derive properties of colimits. Notably, we prove that the forgetful functor from a coslice creates colimits over trees. We also use the construction to examine how colimits interact with orthogonal factorization systems and with cohomology theories. As a consequence of their interaction with orthogonal factorization systems, all pointed colimits (special kinds of coslice colimits) preserve n-connectedness, which implies that higher groups are closed under colimits on directed graphs. We have formalized our main construction of the colimit functor in Agda.

2012 ACM Subject Classification Theory of computation → Type theory

Keywords and phrases colimits, homotopy type theory, category theory, higher inductive types, synthetic homotopy theory

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

Acknowledgements We'd like to thank the anonymous reviewer for HoTT/UF 2023 who pointed out the relationship between adjunctions and factorization systems.

1 Introduction

Homotopy type theory (HoTT) extends Martin-Löf type theory (MLTT) with univalence and higher inductive types [23]. The key feature of HoTT is that all types behave as homotopy types of topological spaces [8]. Thus, with HoTT, we can use purely type-theoretic methods to prove new properties of spaces. Moreover, higher inductive types (HITs) let us bring a huge range of spaces into HoTT. As a result, HoTT is a useful system for developing synthetic homotopy theory and formalizing it in proof assistants like Agda and Coq [7, 5].

We study HITs arising as *(homotopy) colimits* in coslices of a universe, called *coslice colimits*. Coslices of a universe are type-theoretic versions of coslice categories. A colimit in a category is an object formed by gluing togther simpler objects in a coherent fashion. The *coherent* requirement ensures that the colimit has a universal property, which reduces proofs about the colimit to proofs about the simpler objects it is built out of. When these objects are spaces, perhaps endowed with extra structure, colimits built out of them find wide use in homotopy theory. For example, the class of HITs we study includes colimits of *pointed spaces*. Such colimits are key to the *Brown representability theorem* [12, Section 1.4.1], which is about homotopy functors on the (∞-)category of pointed connected spaces. Indeed, the proof relies on the fact that this category is generated under colimits by compact cogroups.

1.1 Contributions

In this section, we explain the contributions of the paper along with its organization. We start by outlining the heart of the paper, which we call *the main connection*. Afterward, we describe its three independent applications in synthetic homotopy theory. Full details and proofs of our development are found in our associated technical report [6]. We have formalized our construction of A-colimits in Agda but not the applications of it. A GitHub repository containing our formalization will appear shortly.

1.1.1 The main connection (Section 5)

Suppose $\mathcal{U}$ is a universe and A is a type in $\mathcal{U}$. We want to construct all colimits in $A/\mathcal{U}$, or *A-colimits*. HoTT has a general schema for HITs that would let us simply postulate A-colimits. We, however, explicitly construct A-colimits with just the machinery of MLTT augmented with pushouts (Section 5).[1] We take this different apporach to reveal the connection between A-colimits and their underlying colimits in $\mathcal{U}$. In fact, our construction is *not* a case of a general method to encode higher-dimensional HITs with pushouts but rather tailored to reveal this connection.

Why do we care about this connection? It sheds light on three established areas of synthetic homotopy theory. We preview them now and will return to them in Sections 6–8.

The universality of colimits (Section 6)

The *universality* of colimits is a special feature of locally cartesian closed (LCC) ∞-categories, such as that of spaces. The main connection will establish a well-known classical result inside type theory: The forgetful functor $A/\mathcal{U} \to \mathcal{U}$ *creates* colimits of diagrams over contractible graphs (Theorem 15).[2] Examples of such colimits include sequential colimits [21]. With the forgetful functor creating colimits, we can transfer the universality of A-colimits in a large number of cases (Theorem 16). This is notable because LCC categories are not closed under coslices.

The categories of higher groups are cocomplete (Section 7)

A striking feature of colimits is their interaction with orthogonal factorization systems. In Section 7, we use the main connection to show that colimits in $A/\mathcal{U}$ preserve left classes of maps of such systems on $\mathcal{U}$. It is significant that we consider systems on $\mathcal{U}$ rather than $A/\mathcal{U}$. We could derive a similar preservation theorem for systems on $A/\mathcal{U}$ directly from the universal property of an A-colimit. In practice, however, the orthogonal factorization systems we tend to care about are on $\mathcal{U}$. Since the main connection relates the action of A-colimits on maps to the action of their underlying colimits on maps (Section 5.4), we manage to deduce the preservation theorem for systems on $\mathcal{U}$.

To prove this theorem, we find it useful to develop the theory of orthogonal factorization systems in a more general setting than $\mathcal{U}$. In Section 4.1, we study such systems on *wild categories*, which make up one approach to category theory in HoTT. We prove that if a functor F of well-behaved wild categories with orthogonal factorization systems has a right adjoint G, then F preserves the left class when G preserves the right class (Theorem 11). We combine this result with the main connection to deduce the desired preservation property.

When we focus on the (n-connected, n-truncated) system on $\mathcal{U}$ [23, Section 7.6] and take A as the unit type, the main connection shows that the colimit of every diagram of pointed n-connected types is n-connected. One useful corollary of this is that the higher category

[1] A theoretical advantage of such a construction is that pushouts, the simplest nontrivial HITs, can be postulated with a less powerful schema than that required to postulate A-colimits.
[2] For a definition of *creating (co)limits*, see [13].

(n,k) GType of k-tuply groupal n-groupoids considered by [2] is cocomplete on (directed) graphs for all truncation levels $-2 \leq n \leq \infty$ and $-1 \leq k < \infty$ (Example 18).

Cohomology sends colimits to weak limits (Section 8)

Finally, we examine how cohomology theories interact with colimits. To do this, we consider *weak limits*, which are key ingredients in the Brown representability theorem (BRT). A weak colimit in a category need not satify the uniqueness property required of a colimit. The BRT specifies conditions for a presheaf on the homotopy category $\mathbf{Ho}(\mathbf{Top}_{*,c})$ of pointed connected spaces to be representable. The known proof of this theorem requires the presheaf to send countable homotopy colimits in $\mathbf{Top}_{*,c}$ to weak limits in $\mathbf{Set}$. Eilenberg-Steenrod cohomology theories enjoy this property as set-valued functors, which may be viewed as the generalized Mayer-Vietoris property of cohomology.

In Section 8, we use the main connection to establish a restricted, type-theoreitc version of this property. From the main connection we derive another construction of A-colimits, as pushouts of coproducts (Corollary 20), which mirrors a well-known classical lemma. We take A as the unit type and combine the new construction with the Mayer-Vietoris sequence to find that cohomology takes finite colimits to weak limits assuming the axiom of choice.

2 Additional related work

2.1 Construction of nonrecursive 2-HITs

The HITs we consider are nonrecursive 2-HITs, in the sense that they have only nonrecursive constructors of points and of paths of dimension one or two. Van Doorn et al. explicitly construct nonrecursive 2-HITs in MLTT augmented with puhsouts [24]. When specialized to A-colimits, however, their construction has a significantly different form from ours and does not directly lead to the properties of A-colimits we derive. Moreover, they do not prove the full induction principle enjoyed by the 2-HIT for their construction, whereas we do for ours. The full induction principle is necessary (and sufficient) to characterize the 2-HIT uniquely.

2.2 Orthogonal factorization systems

Our work also builds on the theory of orthogonal factorization systems. Such systems play important roles in model category theory [15], a key framwork for classical homotopy theory. Moreover, in type theory, Rijke et al. have shown that such systems on $\mathcal{U}$ are closely connected to modalites [18], which are important in logic. We extend such systems to categories other than $\mathcal{U}$. Moreover, we lift such systems on $\mathcal{U}$ to categories of diagrams in $\mathcal{U}$ (Lemma 17).

3 Background on type theory and colimits

Before describing the main connection and its applications, we need to review the type system we work in. For us, the most important data type of this system is the colimit of a diagram of types over a graph, or the *ordinary colimit.*

3.1 Type system

We assume the reader is familiar with MLTT and HITs in the style of [23]. We will work in MLTT augmented with ordinary colimits, nonrecursive 1-HITs defined in Section 3.3. In fact, we need only augment MLTT with pushouts as they let us construct all nonrecursive

1-HITs with all of their computational properties. Our system, denoted by $\mathsf{MLTT} + \mathsf{Colim}$, gives us strong function extensionality for free. Some applications of the main connection also use Voevodsky's univalence axiom. We will note each time that we use univalence.

3.2 Graphs

Let $\mathcal{U}$ be a universe and $A : \mathcal{U}$. In classical category theory, a diagram is a functor $\mathcal{I} \to \mathcal{C}$ of categories. As long as $\mathcal{C}$ is cocomplete, we can form the functor $\mathsf{colim}_{\mathcal{I}}$ sending each diagram over $\mathcal{I}$ to its colimit in $\mathcal{C}$. We, however, want the colimit of a diagram in the ∞-category $A/\mathcal{U}$, known as a homtopy colimit. This requires the diagram to be homotopy coherent at infinitely many levels. It is unknown whether one can define such diagrams in HoTT.

Still, we can define fully coherent diagrams over free categories on a graph. A *graph* is a pair $\Gamma :\equiv (\Gamma_0, \Gamma_1)$ consisting of a type $\Gamma_0 : \mathcal{U}$ of vertices and a family $\Gamma_1 : \Gamma_0 \to \Gamma_0 \to \mathcal{U}$ of edges. A Γ*-shaped diagram in* $A/\mathcal{U}$ is a pair $F :\equiv (F_0, F_1)$ consisting of a function $F_0 : \Gamma_0 \to A/\mathcal{U}$ and a family of maps $F_1 : (i, j : \Gamma_0) \to \Gamma_1(i, j) \to F_0(i) \to_A F_0(j)$. Here, we have defined $A/\mathcal{U} :\equiv \sum_{T:\mathcal{U}} A \to T$ and $X \to_A Y :\equiv \sum_{k:\mathsf{pr}_1(X)\to\mathsf{pr}_1(Y)} k \circ \mathsf{pr}_2(X) \sim \mathsf{pr}_2(Y)$ where $X, Y : A/\mathcal{U}$. We may write F for F_0 and F_1. We remark that F is implicitly coherent because its domain is freely generated by the points and edges of Γ.

▶ **Example 1.** For each graph Γ and $D : \mathsf{Ob}(A/\mathcal{U})$, the *constant diagram* $\mathsf{const}_\Gamma(D)$ *at* D is defined by $(\mathsf{const}_\Gamma(D))_0\,(i) :\equiv D$ and $(\mathsf{const}_\Gamma(D))_1\,(i, j, g) :\equiv \mathsf{id}_D$. We often refer to $\mathsf{const}_\Gamma(D)$ simply by D.

We will see that A-colimits interact nicely with *trees*. A tree is a graph without cycles. Formally, a graph Γ is a *tree* if the quotient Γ_0/Γ_1 is contractible. Both $\mathbb{N}$ and $\mathbb{Z}$ are trees when equipped with the successor ordering:

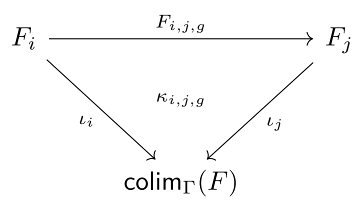
$$\mathbb{N} \;\equiv\; 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots \qquad \mathbb{Z} \;\equiv\; \cdots \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow \cdots$$

Rijke has defined the notion of *directed tree* and has defined an interpretation function sending an element of a W-type to a directed tree [19, "The underlying trees of elements of W-types"]. Every directed tree is a tree in our sense (see [6, Lemma 2.0.8]). Thus, every element of a W-type can be realized as a tree.

3.3 Colimits in $\mathcal{U}$

Let F be a Γ-shaped diagram in $\mathcal{U}$. The *colimit of* F is the HIT $\mathsf{colim}_\Gamma(F)$ generated by

$$\begin{aligned} \iota &: (i : \Gamma_0) \to F_i \to \mathsf{colim}_\Gamma(F) \\ \kappa &: (i, j : \Gamma_0)\,(g : \Gamma_1(i, j)) \to \iota_j \circ F_{i,j,g} \sim \iota_i \end{aligned}$$



What characterizes $\mathsf{colim}_\Gamma(F)$ as a colimit of F is that κ is *(homotopy) initial* in the category of cocones under F (or F-cocones) [20]. Equivalently, for every $X : \mathcal{U}$, the function

$$\begin{aligned} \mathsf{postcomp} &: (\mathsf{colim}_\Gamma(F) \to X) \;\to\; \mathsf{Cocone}_F(X) \\ \mathsf{postcomp}(f) &:\equiv \big(\lambda i. f \circ \iota_i, \lambda i \lambda j \lambda g \lambda (x : F_i). \mathsf{ap}_f(\kappa_{i,j,g}(x))\big) \end{aligned}$$

is an equivalence, where $\mathsf{Cocone}_F(X)$ denotes the type of F-cocones on X.

4 Wild categories

The coslice of a universe fits into the framework of *wild categories*. This is one approach to category theory in HoTT and is used by other works of synthetic homotopy theory [10, 3, 5]. It provides us an interface for key definitions and lemmas throughout the paper. It is especially useful for the relationship between A-colimits and orthogonal facotrization systems we establish in Section 7. This relationship requires us to formulate orthogonal factorization systems on categories other than universes, namely the category of type-valued diagrams over a graph.

The key distinction between wild categories and (pre-)categories [23, Section 9.1] is that the latter have 0-truncated hom types. This means that instead of cutting away the higher coherence data for morphisms, wild categories simply ignore them. We choose them over pre-categories because we will focus on type-theoretic universes and their coslices (see Example 6), which are wild categories but not pre-categories in general.

▶ **Definition 2** ([6, Definition 3.1.1]). *A wild category (in a universe $\mathcal{U}$) is a tuple $\mathcal{C}$ consisting of a type $\mathsf{Ob} : \mathcal{U}$ of objects, a family $\mathsf{hom}_{\mathcal{C}} : \mathsf{Ob} \to \mathsf{Ob} \to \mathcal{U}$ of hom types, idenity morphisms id, composition $\circ$, left LId and right RId unit laws, and associativity laws assoc.*

By itself, the data of a wild category is insufficient for our work on orthogonal factorization systems. We need two extra ingredients. The first is the data of a bicategory, which is defined as in classical 2-category theory. The second is a wild-categorical version of univalence.

▶ **Definition 3.** *A wild category $\mathcal{C}$ is a bicategory if it is equipped with identities*
(a) $\mathsf{ap}_{-\circ f}(\mathsf{assoc}(k, g, h)) \cdot \mathsf{assoc}(k, g \circ h, f) \cdot \mathsf{ap}_{k\circ -}(\mathsf{assoc}(g, h, k)) =$
$\mathsf{assoc}(k \circ g, h, f) \cdot \mathsf{assoc}(k, g, h \circ f)$ *for all composable morphisms k, g, h, and f*
(b) $\mathsf{assoc}(g, \mathsf{id}, h) \cdot \mathsf{ap}_{g\circ -}(\mathsf{LId}(h)) = \mathsf{ap}_{-\circ h}(\mathsf{RId}(g))$ *for all composable morphisms g and h.*[3]

▶ Remark. For us, a bicategory is always a $(2, 1)$-category since the 2-cells, which are identities in $\mathcal{U}$, are invertible.

Before moving to univalence, we transfer a well-known lemma of classical 2-category theory to type theory. This was first proved for monoidal categories [9], but the proof is applicable to all bicategories. (The type-theoretic version also appears as [3, Lemma 4.3].)

▶ **Lemma 4** ([6, Lemma 3.1.6]). *Let $\mathcal{C}$ be a bicategory. For all $A, B, C : \mathsf{Ob}(\mathcal{C})$, $f : \mathsf{hom}_{\mathcal{C}}(A, B)$, $g : \mathsf{hom}_{\mathcal{C}}(B, C)$, we have $\mathsf{LId}(g \circ f)^{-1} \cdot \mathsf{assoc}(\mathsf{id}, g, f)^{-1} \cdot \mathsf{ap}_{-\circ f}(\mathsf{LId}(g)) \;=\; \mathsf{refl}_{g\circ f}$.*

▶ **Definition 5.** *Let $\mathcal{C}$ be a bicategory. We say that $\mathcal{C}$ is univalent if the canonical function $\big(A =_{\mathsf{Ob}(\mathcal{C})} B\big) \to (A \simeq_{\mathcal{C}} B)$ is an equivalence. Here, elements of the righthand type are equivalences, defined as bi-invertible morphisms (in the manner of [23, Definition 4.3.1]).*

▶ **Example 6.** The following are univalent bicategories assuming the univalence axiom.

- The category $\mathcal{U}$ of types and functions
- For each $A : \mathcal{U}$, the coslice $A/\mathcal{U}$ of $\mathcal{U}$ under A
- The category $\mathsf{Diag}(\Gamma, A/\mathcal{U})$ of Γ-shaped diagrams in $A/\mathcal{U}$. We define its hom types (natural transformations) when we present the action of the A-colimit on maps (Section 5.4).

Our ultimate interest is in colimits in the wild category $A/\mathcal{U}$. This category is defined by

$$\mathsf{Ob}(A/\mathcal{U}) \;:\equiv\; \sum_{X:\mathcal{U}} A \to X \qquad \mathsf{hom}_{A/\mathcal{U}}(X, Y) \;:\equiv\; X \to_A Y$$

[3] A wild bicategory is called a *wild 2-precategory* by [3].

For each $X : \mathsf{Ob}(A/\mathcal{U})$, the identity morphism on X is $\big(\mathsf{id}_{\mathsf{pr}_1(X)}, \lambda a.\mathsf{refl}_{\mathsf{pr}_2(X)(a)}\big)$. Composition is defined by $(g, g_p) \circ (f, f_p) :\equiv \big(g \circ f, \lambda a.\, \mathsf{ap}_g(f_p(a)) \cdot g_p(a)\big)$. The associativity and unit laws follow from routine path algebra. Note the categories $\mathbf{0}/\mathcal{U}$ and $\mathcal{U}$ are equivalent.

We write ty and str for the functions $\mathsf{pr}_1 : \mathsf{Ob}(A/\mathcal{U}) \to \mathcal{U}$ and $\mathsf{pr}_2 : (Z : \mathsf{Ob}(A/\mathcal{U})) \to A \to \mathsf{pr}_1(Z)$, respectively. Also, we write fun and pt for the functions $\mathsf{pr}_1 : \mathsf{hom}_{A/\mathcal{U}}(W, Z) \to \mathsf{ty}(W) \to \mathsf{ty}(Z)$ and $\mathsf{pr}_2 : \big(h : \mathsf{hom}_{A/\mathcal{U}}(W, Z)\big) \to \mathsf{pr}_1(h) \circ \mathsf{str}(W) \sim \mathsf{str}(Z)$, respectively.

▶ **Lemma 7.** *Let $f, g : X \to_A Y$ and $H : \mathsf{fun}(f) \sim \mathsf{fun}(g)$. Let $f \sim_A g :\equiv (a : A) \to H(\mathsf{str}(X)(a))^{-1} \cdot \mathsf{pt}(f)(a) = \mathsf{pt}(g)(a)$. We have an equivalence $\langle H, -\rangle : f \sim_A g \to f = g$.*

4.1 Orthogonal factorization systems

We now introduce orthogonal factorization systems on wild categories. For us, the key property of such systems is that they interact nicely with adjunctions. In Section 7, we deduce from this property, combined with the main connection, that A-colimits preserve the left classes of orthogonal factorization systems on $\mathcal{U}$.

▶ **Definition 8.** *Let $\mathcal{C}$ be a wild category. An* orthogonal factorization system (OFS) *on $\mathcal{C}$ consists of predicates $\mathcal{L}, \mathcal{R} : \prod_{A,B:\mathcal{C}} \mathsf{hom}_{\mathcal{C}}(A, B) \to \mathsf{Prop}$ such that*
1. *both $\mathcal{L}$ and $\mathcal{R}$ are closed under composition and have all identities;*
2. *for every $h : \mathsf{hom}_{\mathcal{C}}(A, B)$, the following type is contractible:*

$$\mathsf{fact}_{\mathcal{L},\mathcal{R}}(h) :\equiv \sum_{D:\mathcal{C}} \sum_{f:\mathsf{hom}_{\mathcal{C}}(A,D)} \sum_{g:\mathsf{hom}_{\mathcal{C}}(D,B)} g \circ f = h \times \mathcal{L}(f) \times \mathcal{R}(g).$$

For the next lemma, where $\mathcal{C}$ is a univalent bicategory, $\mathcal{C}$ is similar enough to $\mathcal{U}$ that the proof for $\mathcal{U}$ can be transfered to $\mathcal{C}$.[4] Indeed, univalence lets us characterize the identity types of $\mathsf{fact}_{\mathcal{L},\mathcal{R}}(h)$ via the fundamental theorem of identity types [17, Theorem 11.2.2]. Moreover, Lemma 4 gives us a suitable diagonal filler for the key commuting square used by the proof.

▶ **Lemma 9** ([6, Corollary 3.2.6])**.** *Suppose that $\mathcal{C}$ is a univalent bicategory with an OFS $(\mathcal{L}, \mathcal{R})$. A map is in $\mathcal{L}$ if and only if it has the left lifitng property against $\mathcal{R}$.*[5]

This alternative definition of $\mathcal{L}$ is useful for the proof of Theorem 11, below. For this theorem, we need to introduce adjoint pairs of functors between wild categories.

▶ **Definition 10.** *Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be functors of wild categories. An adunction $L \dashv R$ consists of an equivalence $\alpha : \mathsf{hom}_{\mathcal{D}}(LA, X) \simeq \mathsf{hom}_{\mathcal{C}}(A, RX)$ for all $A : \mathsf{Ob}(\mathcal{C})$ and $X : \mathsf{Ob}(\mathcal{D})$ along with naturality proofs:*

$$\mathsf{n}_1 : (A : \mathsf{Ob}(\mathcal{C}))\,(X, Y : \mathsf{Ob}(\mathcal{D}))\,(g : \mathsf{hom}_{\mathcal{D}}(X, Y))\,(h : \mathsf{hom}_{\mathcal{D}}(LA, X)) \to R(g) \circ \alpha(h) = \alpha(g \circ h)$$
$$\mathsf{n}_2 : (Y : \mathsf{Ob}(\mathcal{D}))\,(A, B : \mathsf{Ob}(\mathcal{C}))\,(f : \mathsf{hom}_{\mathcal{C}}(A, B))\,(h : \mathsf{hom}_{\mathcal{D}}(LB, Y)) \to \alpha(h) \circ f = \alpha(h \circ L(f)).$$

▶ **Theorem 11** ([6, Corollary 3.2.9])**.** *Consider an adjunction $L \dashv R$ where both $\mathcal{C}$ and $\mathcal{D}$ are univalent bicategories. If R preserves $\mathcal{R}$, then L preserves $\mathcal{L}$.*

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[4] For the proof of this lemma for $\mathcal{U}$, see [18, Lemma 1.46].
[5] The left lifting property is defined in [6, Definition 3.2.3].

5 The main connection

Let Γ be a graph and suppose F is a diagram in $A/\mathcal{U}$ over Γ. We want to construct the A-*colimit* of F in $\mathsf{MLTT} + \mathsf{Colim}$ so as to show the connection between A-colimits and ordinary colimits. After defining an A-colimit of F, we mention a reasonable yet wrong approach to constructing it. Then, we explain another construction and prove it is correct by exhibiting it as left adjoint to the constant diagram functor

5.1 Definition of A-colimits

We can generalize ordinary colimits in Section 3 to all coslices $A/\mathcal{U}$. For each $Y : \mathsf{Ob}(A/\mathcal{U})$, an F-cocone on Y consists of a family of maps $h : (i : \Gamma_0) \to F_i \to_A Y$ in $A/\mathcal{U}$ together with an identity $H_{i,j,g} \ : \ h_j \circ F_{i,j,g} = h_i$ for all $i, j : \Gamma_0$ and $g : \Gamma_1(i, j)$. In this situation, we say that Y is a *colimit of* F if (h, H) is iniital in the category of F-cocones. This means that for each $X : \mathsf{Ob}(A/\mathcal{U})$, the function

$$\begin{aligned} \mathsf{postcomp}(h, H) \ &: \ (Y \to_A X) \ \to \ \mathsf{Cocone}_F(X) \\ \mathsf{postcomp}(h, H, f) \ &:= \ \big(\lambda i. f \circ h_i, \lambda i \lambda j \lambda g. \mathsf{assoc}(f, h_j, F_{i,j,g}) \cdot \mathsf{ap}_{f \circ -}(H_{i,j,g})\big) \end{aligned}$$

is an equivalence. We must include the associativy term since associativity of maps does not hold judgmentally in $A/\mathcal{U}$ (whereas it does in $\mathcal{U}$).

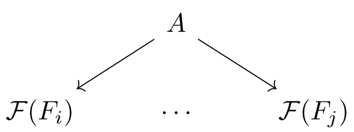
Observe that by a variant of Lemma 7, $h_j \circ F_{i,j,g} = h_i$ is equivalent to the type of homotopies $\eta_{i,j,g} : \mathsf{fun}(h_j) \circ \mathsf{fun}(F_{i,j,g}) \sim \mathsf{fun}(h_i)$ equipped with a path

$$\eta(\mathsf{str}(F_i)(a)) \ = \ \mathsf{ap}_{\mathsf{fun}(h_j) \circ \mathsf{fun}(F_{i,j,g})}(\mathsf{pt}(F_{i,j,g})(a)) \cdot \mathsf{pt}(h_j)(a) \cdot \mathsf{pt}(h_i)(a)^{-1} \qquad (2\text{-}\mathsf{c})$$

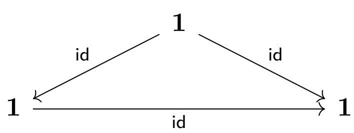
for each $a : A$. It is this family of 2-cells which distinguishes the colimit of F, in $A/\mathcal{U}$, from $\mathsf{colim}_\Gamma(\mathcal{F}(F))$. Here, we reuse $\mathcal{F}$ to denote the evident forgetful functor from Γ-shaped diagrams in $A/\mathcal{U}$ to those in $\mathcal{U}$. The 2-cells affect $\mathsf{colim}_\Gamma(\mathcal{F}(F))$ by collapsing its nontrivial loops formed by paths of the form $\eta(\mathsf{str}(F_i)(a))$. We call such loops *distinguished loops* in $\mathsf{colim}_\Gamma(\mathcal{F}(F))$. For example, if $i \equiv j$ and $F_{i,j,g} \equiv \mathsf{id}_{F_i}$, then (2-c) is equivalent to $\eta(\mathsf{str}(F_i)(a)) = \mathsf{refl}_{\mathsf{fun}(h_i)(\mathsf{str}(F_i)(a))}$. In this case, a 2-cell fills $\eta(\mathsf{str}(F_i)(a))$.

5.2 Misleading approach

If our setting behaved like the classical one, the colimit of F in $A/\mathcal{U}$ would arise as the ordinary colimit of $\mathcal{F}(F)$ augmented with the canonical arrow from A to $\mathcal{F}(F_i)$ for each $i : \Gamma_0$ [14, Proposition 4.6]. If Γ is discrete, i.e., Γ_1 is the empty relation, then the A-colimit of F inside HoTT is, in fact, the colimit of



In general, though, this construction is wrong inside HoTT. For example, the pointed colimit of the diagram $\mathbf{1} \xrightarrow{\mathsf{id}} \mathbf{1}$ is trivial, but the colimit of the augmented diagram

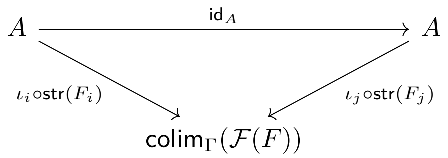


is the circle S^1. The reason for the discrepancy between the classical case and ours is that unless Γ is discrete, the augmented diagram inside HoTT adds arrows that are intended as composites but are not interpreted as such in the model of HoTT. Indeed, the model sees them as freely added to the diagram.

5.3 Our approach

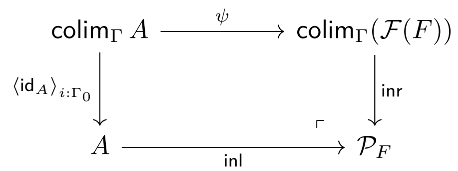
Our approach to building the colimit of F never creates an augmented diagram, thereby avoiding the problem of Section 5.2. We start with the colimit $\mathsf{colim}_\Gamma(\mathcal{F}(F))$ which ignores the coslice structure of F. Then, we glue onto this colimit the 2-cells required by the coslice colimit. We do this via a quotient of $\mathsf{colim}_\Gamma(\mathcal{F}(F))$ that fills its distinguished loops.

To this end, define $\mathsf{colim}_\Gamma A \xrightarrow{\psi} \mathsf{colim}_\Gamma(\mathcal{F}(F))$ by colimit induction, as the function induced by the cocone

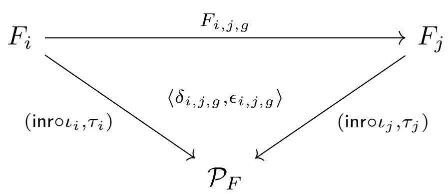


$$(a \mapsto \mathsf{ap}_{\iota_j}(\mathsf{pt}(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{str}(F_i)(a)))$$

under the constant diagram at A. Intutively, this map finds the distinguished loops of $\mathsf{colim}_\Gamma(\mathcal{F}(F))$. Next, form the pushout square



With the equivalence $\langle -, - \rangle$ of Lemma 7, we can form an F-cocone on $(\mathcal{P}_F, \mathsf{inl})$



$$(\tau_i(a) :\equiv \mathsf{glue}_{\mathcal{P}_F}(\iota_i(a))^{-1})$$

as follows. We have a homotopy $\delta_{i,j,g} :\equiv \lambda(x : \mathsf{ty}(F_i)).\, \mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(x))$ from $\mathsf{inr} \circ \iota_j \circ \mathsf{fun}(F_{i,j,g})$ to $\mathsf{inr} \circ \iota_i$. Further, for each $a : A$, we have a chain $\epsilon_{i,j,g}(a)$ of identities

$$\begin{aligned}
& \mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{str}(F_i)(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pt}(F_{i,j,g})(a)) \cdot \tau_j(a) \\
=\ & \mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\mathsf{pt}(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{str}(F_i)(a)))^{-1} \cdot \tau_j(a) \cdot \mathsf{refl}_{\mathsf{inl}(a)} \\
=\ & \mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\psi}(\kappa_{i,j,g}(a)))^{-1} \cdot \tau_j(a) \cdot \mathsf{refl}_{\mathsf{inl}(a)} && (\textit{via } \rho_\psi(i,j,g,a)) \\
=\ & \mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\psi}(\kappa_{i,j,g}(a)))^{-1} \cdot \tau_j(a) \cdot \mathsf{ap}_{\mathsf{inl}}(\mathsf{ap}_{\langle \mathsf{id}_A \rangle}(\kappa_{i,j,g}(a))) && (\textit{via } \rho_{\langle \mathsf{id}_A \rangle}(i,j,g,a)) \\
=\ & (\kappa_{i,j,g}(a))_*(\tau_j(a)) && (\textit{transport on identity type}) \\
=\ & \tau_i(a) && (\textit{by } \mathsf{apd}_{\mathsf{glue}(-)^{-1}}(\kappa_{i,j,g}(a)))
\end{aligned}$$

Let $\mathcal{K}(\mathcal{P}_F)$ denote this F-cocone structure on $(\mathcal{P}_F, \mathsf{inl})$.

▶ **Theorem 12** ([6, Theorem 4.4.3])**.** *The function*

$$\mathsf{postcomp}(\mathcal{K}(\mathcal{P}_F), T, f_T) \;:\; ((\mathcal{P}_F, \mathsf{inl}) \to_A (T, f_T)) \;\to\; \mathsf{Cocone}_F(T, f_T)$$

is an equivalence for every $(T, f_T) : \mathsf{Ob}(A/\mathcal{U})$*.*

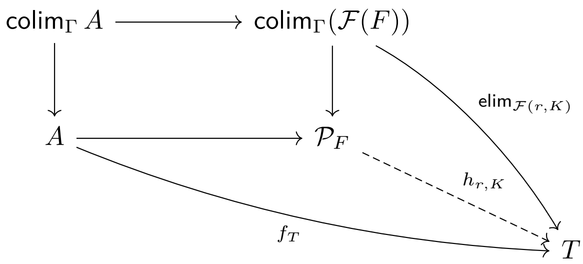
Proof. We construct a quasi-inverse $\mathsf{Cocone}(T, f_T) \xrightarrow{\Theta} ((\mathcal{P}_F, \mathsf{inl}) \to_A (T, f_T))$ of $\mathsf{postcomp}$ as follows. Let $(r, K) : \mathsf{Cocone}(T, f_T)$. The forgetful functor $\mathcal{F}$ from cocones under F to ordinary cocones under $\mathcal{F}(F)$ gives rise to the function $\mathsf{elim}_{\mathcal{F}(r,K)} \;:\; \mathsf{colim}_\Gamma(\mathcal{F}(F)) \to T$ by the colimit elimination principle. For all $i : \Gamma_0$ and $a : A$, we have

$$f_T(a) \xlongequal{\mathsf{pt}(r_i)(a)^{-1}} \mathsf{fun}(r_i)(\mathsf{str}(F_i(a))) \xlongequal{\quad} \mathsf{elim}_{\mathcal{F}(r,K)}(\mathsf{str}(F_i(a)))$$

Further, for all $i, j : \Gamma_0$, $g : \Gamma_1(i, j)$, and $a : A$, we have a chain of identities

$$\begin{aligned}
&\mathsf{transp}^{x \mapsto f_T([\mathsf{id}_A](x)) = \mathsf{rec}_{\mathsf{colim}}(U_F(r,K))(\psi(x))}(\kappa_{i,j,g}(a), \mathsf{pr}_2(r_j)(a)^{-1}) \\
&= \mathsf{ap}_{f_T}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a)))^{-1} \cdot \mathsf{pr}_2(r_j)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(U_F(r,K))}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))) \\
&\hfill (\textit{transport on identity type}) \\
&= \mathsf{ap}_{f_T}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a)))^{-1} \cdot \mathsf{pr}_2(r_j)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pt}(F_{i,j,g})(a))^{-1} \cdot \mathsf{pr}_1(K_{j,i,g})(\mathsf{str}(F_i)(a)) \\
&\hfill (\textit{via } \rho_\psi(i,j,g,a) \textit{ and then } \rho_{\mathsf{elim}_{\mathcal{F}(r,K)}}(i,j,g,\mathsf{str}(F_i)(a))) \\
&= \Big(\mathsf{pr}_1(K_{j,i,g})(\mathsf{str}(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pt}(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)\Big)^{-1} \quad (\textit{via } \rho_{[\mathsf{id}_A]}(i,j,g,a)) \\
&= \mathsf{pr}_2(r_i)(a)^{-1} \qquad (\textit{by } \mathsf{ap}_{_^{-1}}(\mathsf{pr}_2(K_{j,i,g})(a)))
\end{aligned}$$

By induction on $\mathsf{colim}_\Gamma A$, this gives us a homotopy $f_T \circ \langle \mathsf{id}_A \rangle \sim \mathsf{elim}_{\mathcal{F}(r,K)} \circ \psi$ and thus a function $h_{r,K} : \mathcal{P}_F \to T$



defined by pushout induction on $\mathcal{P}_F$. Finally, since $h(\mathsf{inl}(a)) \equiv f_T(a)$ for all $a : A$, we have

$$\Theta(r, K) :\equiv \big(h_{r,K}, \lambda a.\mathsf{refl}_{f_T(a)}\big) \;:\; (\mathcal{P}_F, \mathsf{inl}) \to_A (T.f_T).$$

Proving that $\mathsf{postcomp}(\mathcal{K}(\mathcal{P}_F), T, f_T) \circ \Theta \sim \mathsf{id}$ and $\Theta \circ \mathsf{postcomp}(\mathcal{K}(\mathcal{P}_F), T, f_T) \sim \mathsf{id}$ involves elaborate computations. We have formalized the proofs of both homotopies. ◀

5.4 Action on maps

So far, we have defined a function $\mathsf{colim}^A_\Gamma :\equiv \mathcal{P} : \mathsf{Ob}(\mathsf{Diag}(\Gamma, A/\mathcal{U})) \to \mathsf{Ob}(A/\mathcal{U})$ sending a Γ-shaped diagram in $A/\mathcal{U}$ to its A-colimit. We now make $\mathcal{P}$ a functor by describing its action on maps of diagrams. We want to describe this action in terms of the action of the ordinary colimit functor by using the special form of $\mathcal{P}$'s object function.

Suppose that F and G are Γ-shaped diagrams in $A/\mathcal{U}$. The type of *natural transformations* from F to G consists of families $\alpha : (i : \Gamma_0) \to \mathsf{ty}(F_i) \to_A \mathsf{ty}(G_i)$ of maps equipped with an

A-homotopy $G_{i,j,g} \circ \alpha_i \sim_A \alpha_j \circ F_{i,j,g}$ for all i, j, g, where $\sim_A$ is as in Lemma 7. Consider a morphism $\delta :\equiv \big(d, \big\langle \xi, \tilde{\xi} \big\rangle\big)$

$$\begin{array}{ccc} F_i & \xrightarrow{F_{i,j,g}} & F_j \\ {\scriptstyle d_i}\big\downarrow & \big\langle \xi_{i,j,g}, \tilde{\xi}_{i,j,g} \big\rangle & \big\downarrow{\scriptstyle d_j} \\ G_i & \xrightarrow[G_{i,j,g}]{} & G_j \end{array}$$

from F to G, where $\langle -, - \rangle$ is as in Lemma 7. We have a commuting square

$$\begin{array}{ccc} F_i & \xrightarrow{d_i} & G_i \\ {\scriptstyle \iota_i^F}\big\downarrow & & \big\downarrow{\scriptstyle \iota_i^G} \\ \mathsf{colim}_\Gamma^A(F) & \xrightarrow[\mathsf{colim}_\Gamma^A(\delta)]{} & \mathsf{colim}_\Gamma^A(G) \end{array} ,$$

obtained as follows. We have $\mathsf{colim}_\Gamma(\mathcal{F}(F)) \xrightarrow{\bar{\delta}} \mathsf{colim}_\Gamma(\mathcal{F}(G))$ induced by the map

$$\begin{array}{ccc} \mathsf{ty}(F_i) & \xrightarrow{\mathsf{fun}(F_{i,j,g})} & \mathsf{ty}(F_j) \\ {\scriptstyle \mathsf{fun}(d_i)}\big\downarrow & \xi_{i,j,g} & \big\downarrow{\scriptstyle \mathsf{fun}(d_j)} \\ \mathsf{ty}(G_i) & \xrightarrow[\mathsf{fun}(G_{i,j,g})]{} & \mathsf{ty}(G_j) \end{array}$$

of diagrams in $\mathcal{U}$ over Γ. Note that for each $a : A$,

$$\tilde{\xi}_{i,j,g}(a) \ : \ \xi_{i,j,g}(\mathsf{str}(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{fun}(G_{i,j,g})}(\mathsf{pt}(d_i)(a)) \cdot \mathsf{str}(G_{i,j,g})(a) = \mathsf{ap}_{\mathsf{fun}(d_j)}(\mathsf{pt}(F_{i,j,g})(a)) \cdot \mathsf{pt}(d_j)(a).$$

We may assume that $\tilde{\xi}_{i,j,g}(a)$ instead has the equivalent type

$$\xi_{i,j,g}(\mathsf{str}(F_i)(a)) = \underbrace{\mathsf{ap}_{\mathsf{fun}(G_{i,j,g})}(\mathsf{pt}(d_i)(a)) \cdot \mathsf{str}(G_{i,j,g})(a) \cdot \mathsf{pt}(d_j)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{fun}(d_j)}(\mathsf{pt}(F_{i,j,g})(a))^{-1}}_{E_{i,j,g}(a)} .$$

Here we abbreviate the right endpoint by $E_{i,j,g}(a)$. Now, the triangle

$$\begin{array}{ccc} & \mathsf{colim}_\Gamma A & \\ {\scriptstyle \psi_F}\swarrow & & \searrow{\scriptstyle \psi_G} \\ \mathsf{colim}_\Gamma(\mathcal{F}(F)) & \xrightarrow[\bar{\delta}]{} & \mathsf{colim}_\Gamma(\mathcal{F}(G)) \end{array} \qquad (\psi\text{-tri})$$

commutes by induction on $\mathsf{colim}_\Gamma A$. Indeed, the computation rules of these functions give us

$$C_i(a) :\equiv \mathsf{ap}_{\iota_i}(\mathsf{pt}(d_i)(a)) \ : \ \bar{\delta}(\psi_F(\iota_i(a))) \ = \ \psi_G(\iota_i(a))$$

for all $i : \Gamma_0$ and $a : A$. Further, with $\Lambda_{i,j,g}(a) :\equiv C_j(a) \cdot \mathsf{ap}_{\psi_G}(\kappa_{i,j,g}(a))$, we have a chain

$$\begin{aligned}
&\left(\kappa_{i,j,g}(a)\right)_* \left(C_j(a)\right) \\
&= \mathsf{ap}_{\bar\delta}(\mathsf{ap}_{\psi_F}(\kappa_{i,j,g}(a)))^{-1} \cdot C_j(a) \cdot \mathsf{ap}_{\psi_G}(\kappa_{i,j,g}(a)) && \textit{(transport on identity type)} \\
&= \left(\mathsf{ap}_{\iota_j}(\xi_{i,j,g}(\mathsf{str}(F_i)(a)))^{-1} \cdot \kappa_{i,j,g}(\mathsf{fun}(d_i)(\mathsf{str}(F_i)(a)))\right)^{-1} \cdot \mathsf{ap}_{\iota_j \circ \mathsf{fun}(d_j)}(\mathsf{pt}(F_{i,j,g})(a)) \cdot \Lambda_{i,j,g}(a) \\
& && \textit{(via } \rho_{\psi_F}(i,j,g,a) \textit{ and then } \rho_{\bar\delta}(i,j,g,\mathsf{str}(F_i)(a))\textit{)} \\
&= \left(\mathsf{ap}_{\iota_j}(E_{i,j,g}(\mathsf{str}(F_i)(a)))^{-1} \cdot \kappa_{i,j,g}(\mathsf{fun}(d_i)(\mathsf{str}(F_i)(a)))\right)^{-1} \cdot \mathsf{ap}_{\iota_j \circ \mathsf{fun}(d_j)}(\mathsf{pt}(F_{i,j,g})(a)) \cdot \Lambda_{i,j,g}(a) \\
& && \textit{(by } \mathsf{ap}_{\left(\mathsf{ap}_{\iota_j}(-)^{-1} \cdot \kappa_{i,j,g}(\mathsf{fun}(d_i)(\mathsf{str}(F_i)(a)))\right)^{-1} \cdot \ldots}(\tilde\xi_{i,j,g}(a))\textit{)} \\
&= C_i(a) && \textit{(via } \rho_{\psi_G}(i,j,g,a)\textit{)}
\end{aligned}$$

of identities for all $i, j : \Gamma_0$, $g : \Gamma_1(i,j)$, and $a : A$, so (ψ-tri) commutes. We now have a map

$$\begin{array}{ccccc}
A & \longleftarrow & \mathsf{colim}_\Gamma A & \longrightarrow & \mathsf{colim}_\Gamma(\mathcal{F}(F)) \\
\Big\downarrow \mathsf{id} & \mathsf{refl}_{[\mathsf{id}](x)} & \Big\downarrow \mathsf{id} & C^{-1} & \Big\downarrow \bar\delta \\
A & \longleftarrow & \mathsf{colim}_\Gamma A & \longrightarrow & \mathsf{colim}_\Gamma(\mathcal{F}(G))
\end{array}$$

of spans, which induces a function $\Psi_\delta : \mathcal{P}_F \to \mathcal{P}_G$ by the universal property of pushouts. Since $\Psi_\delta(\mathsf{inl}(a)) \equiv \mathsf{inl}(a)$ for all $a : A$, we may take $\mathsf{colim}^A_\Gamma(\delta)$ as $\left(\Psi_\delta, \lambda a.\mathsf{refl}_{\mathsf{inl}(a)}\right) : \mathcal{P}_F \to_A \mathcal{P}_G$.

To verify that the functor $\mathsf{Diag}(\Gamma, A/\mathcal{U}) \xrightarrow{\mathsf{colim}^A_\Gamma} A/\mathcal{U}$ we've defined is correct, we must show that it is left adjoint to the constant diagram functor, i.e., the universal property of the colimit functor. Specifcally, we must construct the terms n_1 and n_2 required by Definition 10.

▶ **Lemma 13** ([6, Lemma 4.4.5]). *For every map $h^* : T \to_A U$, the following square commutes:*

$$\begin{array}{ccc}
\mathsf{colim}^A_\Gamma(F) \to_A T & \xrightarrow{h^* \circ -} & \mathsf{colim}^A_\Gamma(F) \to_A U \\
\Big\downarrow \mathsf{postcomp}_{F,T} & & \Big\downarrow \mathsf{postcomp}_{F,U} \\
\mathsf{Cocone}_F(T) & \xrightarrow[\mathsf{Cocone}_F(h^* \circ -)]{} & \mathsf{Cocone}_F(U)
\end{array}$$

▶ **Lemma 14** ([6, Lemma 4.4.12]). *For every $T : \mathsf{Ob}(A/\mathcal{U})$ and $\delta : F \Rightarrow_A G$, the following square commutes:*

$$\begin{array}{ccc}
\mathsf{colim}^A_\Gamma(G) \to_A T & \xrightarrow{- \circ \mathsf{colim}^A_\Gamma(\delta)} & \mathsf{colim}^A_\Gamma(F) \to_A T \\
\Big\downarrow \mathsf{postcomp}_{G,T} & & \Big\downarrow \mathsf{postcomp}_{F,T} \\
\mathsf{Cocone}_G(T) & \xrightarrow[\mathsf{Cocone}(T)(- \circ \delta)]{} & \mathsf{Cocone}_F(T)
\end{array}$$

The two lower horizontal functions are induced by post-composition with f^* and pre-composition with δ [6, Definition 4.4.11], respectively.

Lemma 13 is a routine path algebra computation, whereas Lemma 14 takes a lot of work. The proof of Lemma 14 is easier for $\mathsf{postcomp}^{-1}_{F,\mathsf{colim}^A_\Gamma(G)}(K(\delta))$ than for $\mathsf{colim}^A_\Gamma(\delta)$, where $K(\delta)$ is the canonical cocone on $\mathcal{P}_G$ under F induced by δ. Therefore, we decide to reduce the goal to an A-homotopy between the two maps. This is easier than our original goal but requires elaborate computations. We have formalized both Lemma 13 and Lemma 14.

6 Creation of colimits

Classically, if $\mathcal{D}$ is an ∞-category, then the forgetful functors of ∞-coslices create $\mathcal{D}$-shaped colimits when $\mathcal{D}$'s core is contractible. Theorem 15 expresses the same result inside HoTT.

▶ **Theorem 15.** *The forgetful functor $A/\mathcal{U} \to \mathcal{U}$ creates colimits over trees.*

Proof. Intuitively, a tree has no cycles, and thus we have no distinguished loops to fill. As a result, coslice colimits over trees look the same as their underlying colimits in $\mathcal{U}$. See [6, Corollary 4.4.6] for a rigorous proof. ◀

▶ Remark. The fact that the forgetful functor $\mathcal{U}^* \to \mathcal{U}$ from pointed types creates *pushouts* appears in the agda-unimath library, though without proof [19, "Pushouts of pointed types"].

Theorem 15 lets us lift powerful features of ordinary colimits to A-colimits. For example, the distinguishing feature of a LCC ∞-category, such as $\mathcal{U}$, is that all of its colimits are *universal*, i.e., pullback-stable. Although the coslice of a LCC category need not be LCC, we now show that all of its colimits over trees are universal.

▶ **Theorem 16.** *For each tree Γ, the colimit $\mathsf{colim}^A_\Gamma(-)$ is universal.*

Proof. By [16, Corollary 3.5.1], the ordinary colimit is pullback-stable. Thus, if Γ is a tree, then Theorem 15 implies that $\mathsf{colim}^A_\Gamma(-)$ is pullback-stable as well. ◀

7 Preservation of left class of an OFS

In this section, we combine our construction of $\mathsf{colim}^A_\Gamma(\delta)$ from Section 5.4 with Theorem 11 to prove that colim^A_Γ always preserves the left class of an OFS on $\mathcal{U}$. We assume the univalence axiom to have access to the the tools of univalent bicategories developed in Section 4.

Let $(\mathcal{L}, \mathcal{R})$ be an OFS on $\mathcal{U}$. For all diagrams $F, G : \mathcal{D}_\Gamma :\equiv \mathsf{Diag}(\Gamma, \mathcal{U})$ and natural transformations $(H, \gamma) : F \Rightarrow G$, define the predicates $\widehat{\mathcal{L}}(H, \gamma) :\equiv (i : \Gamma_0) \to \mathcal{L}(H_i)$ and $\widehat{\mathcal{R}}(H, \gamma) :\equiv (i : \Gamma_0) \to \mathcal{R}(H_i)$.

▶ **Lemma 17** ([6, Theorem 6.0.7])**.** *Let $Q : F \Rightarrow G$. The following type is contractible:*

$$\mathsf{fact}_{\widehat{\mathcal{L}},\widehat{\mathcal{R}}}(Q) \;:\equiv\; \sum_{M:\mathcal{D}_\Gamma} \sum_{S:F\Rightarrow M} \sum_{T:M\Rightarrow G} (T \circ S = Q) \times \widehat{\mathcal{L}}(S) \times \widehat{\mathcal{R}}(T).$$

By Lemma 17, we see that $(\mathcal{L}, \mathcal{R})$ on $\mathcal{U}$ lifts levelwise to $\mathcal{D}_\Gamma$. Since the functor $\mathsf{const}_\Gamma : \mathcal{U} \to \mathcal{D}_\Gamma$ clearly takes $\mathcal{R}$ to $\widehat{\mathcal{R}}$, we deduce that $\mathsf{colim}_\Gamma(-)$ takes $\widehat{\mathcal{L}}$ to $\mathcal{L}$ by Theorem 11.[6] Now, for each $X, Y : \mathsf{Ob}(A/\mathcal{U})$, consider the predicate $\mathcal{L}_A(f, p) :\equiv \mathcal{L}(f)$ on $X \to_A Y$. Then the functor colim^A_Γ takes $\widehat{\mathcal{L}}_A$ to $\mathcal{L}_A$. Indeed, consider a map $\delta : \mathcal{A} \Rightarrow_A \mathcal{B}$ of A-diagrams. The underlying function of $\mathsf{colim}^A_\Gamma(\delta)$ is induced by the morphism

$$\begin{array}{ccccc} A & \longleftarrow & \mathsf{colim}_\Gamma A & \longrightarrow & \mathsf{colim}_\Gamma(\mathcal{F}(\mathcal{A})) \\ {\scriptstyle \mathsf{id}}\downarrow & & \downarrow{\scriptstyle \mathsf{id}} & & \downarrow{\scriptstyle \bar{\delta}} \\ A & \longleftarrow & \mathsf{colim}_\Gamma A & \longrightarrow & \mathsf{colim}_\Gamma(\mathcal{F}(\mathcal{B})) \end{array}$$

of spans. Thus, if δ is in $\widehat{\mathcal{L}}_A$, then all three vertical functions are in $\mathcal{L}$. Since a map of spans is a map of diagrams, we see that $\mathsf{colim}^A_\Gamma(\delta)$ is in $\mathcal{L}_A$.

[6] The adjunction $\mathsf{colim}_\Gamma \dashv \mathsf{const}_\Gamma$ follows directly from the equivalence $\mathsf{postcomp}$ of Section 3.3.

In particular, if F is a diagram of pointed types over Γ such that each $\mathsf{ty}(F_i)$ is $(\mathcal{L},\mathcal{R})$-connected, then the type $\mathsf{colim}^*_\Gamma(F) :\equiv \mathsf{colim}^\mathbf{1}_\Gamma(F)$ is also $(\mathcal{L},\mathcal{R})$-connected.[7] Indeed, we can deduce that $\mathsf{colim}^*_\Gamma\,\mathbf{1} = \mathbf{1}$ from the construction of $\mathcal{P}_F$. Thus, colim^*_Γ takes the unique map $F \Rightarrow_* \mathbf{1}$ of pointed diagrams to $(c, c_p) : \mathsf{colim}^*_\Gamma(F) \to_* \mathsf{colim}^*_\Gamma\,\mathbf{1}$ where $c : \mathsf{colim}^*_\Gamma(F) \to \mathbf{1}$ is the constant map and $\mathcal{L}(c)$ holds.

▶ **Example 18** ([6, Section 6.1])**.** For each truncation level n, if each $\mathsf{ty}(F_i)$ is n-connected, then so is the underlying type of $\mathsf{colim}^*_\Gamma(F)$. Now, let $-1 \le k < \infty$ also be a truncation level. Recall from [2] the category $(n,k)\,\mathsf{GType}$ of *k-tuply groupal n-groupoids*. This is equivalent to the full subcategory $\mathcal{U}^*_{\ge k, \le n+k}$ of $\mathcal{U}^*$ on $(k-1)$-connected, $(n+k)$-truncated pointed types. As the truncation functor $\|{-}\|_m : A/\mathcal{U} \to A/\mathcal{U}$ creates colimits for each m [6, Corollary 4.2.5], our preceding argument implies that $\mathcal{U}^*_{\ge k, \le n+k}$ has all colimits over graphs.

It is a special feature of *pointed* colimits that they always preserve n-connectedness. Indeed, if Γ is not a tree, then colim_Γ may fail to preserve n-connectedness (see [6, Example 6.0.9(b)]).

8 Mapping colimits to weak limits

Finally, we look at the interaction between colimits and (Eilenberg-Steenrod) cohomology theories. Specifically, we apply the 3×3 lemma to the main connection to obtain the familiar construction of $\mathsf{colim}^A_\Gamma(F)$ as a pushout of coproducts in $A/\mathcal{U}$. Afterward, we apply this new construction to the Mayer-Vietoris sequence to prove that cohomology theories send finite colimits to weak limits in **Set** assuming the axiom of choice.

8.1 Decomposition of A-colimits into simpler pieces

To make use of the 3×3 lemma, we first form the following grid of commuting squares:

$$\begin{array}{ccccc}
\sum_{i,j,g} \mathsf{ty}(F_i) & \xleftarrow{\ \mathsf{id}\ +\ \mathsf{id}\ } & \left(\sum_{i,j,g} \mathsf{ty}(F_i)\right) + \left(\sum_{i,j,g} \mathsf{ty}(F_i)\right) & \xrightarrow{(i,x)\ +\ (j,\mathsf{fun}(F_{i,j,g})(x))} & \sum_i \mathsf{ty}(F_i) \\
\uparrow{\scriptstyle (i,j,g,\mathsf{str}(F_i)(a))} & & \uparrow{\scriptstyle (i,j,g,\mathsf{str}(F_i)(a))\ +\ (i,j,g,\mathsf{str}(F_i)(a))} & & \uparrow{\scriptstyle (i,\mathsf{str}(F_i)(a))} \\
\left(\sum_{i,j} \Gamma_1(i,j)\right) \times A & \xleftarrow{\ \mathsf{id}\ +\ \mathsf{id}\ } & \left(\left(\sum_{i,j} \Gamma_1(i,j)\right) \times A\right) + \left(\left(\sum_{i,j} \Gamma_1(i,j)\right) \times A\right) & \xrightarrow{(i,a)\ +\ (j,a)} & \Gamma_0 \times A \\
\downarrow{\scriptstyle \mathsf{pr}_2} & & \downarrow{\scriptstyle \mathsf{pr}_2\ +\ \mathsf{pr}_2} & & \downarrow{\scriptstyle \mathsf{pr}_2} \\
A & \xleftarrow{\ \mathsf{id}_A\ } & A & \xrightarrow{\ \mathsf{id}_A\ } & A
\end{array}$$

Call the pushouts of the left, middle, and right vertical spans V_1, V_2, and V_3, respectively. Call the pushouts of the top, middle, and bottom horizontal spans H_1, H_2, and H_3, respectively. We can form two additional pushouts from this grid:

$$\begin{array}{ccc}
V_2 & \xrightarrow{\ \delta_2\ } & V_3 \\
{\scriptstyle \delta_1}\downarrow & & \downarrow{\scriptstyle \mathsf{inr}} \\
V_1 & \xrightarrow[\ \mathsf{inl}\ ]{} & P_V
\end{array}
\qquad\qquad
\begin{array}{ccc}
H_2 & \xrightarrow{\ \eta_1\ } & H_1 \\
{\scriptstyle \eta_2}\downarrow & & \downarrow{\scriptstyle \mathsf{inr}} \\
H_3 & \xrightarrow[\ \mathsf{inl}\ ]{} & P_H
\end{array}$$

- δ_1 denotes the function induced by the middle-to-left map of spans;

[7] A type X is $(\mathcal{L},\mathcal{R})$-connected if the function $X \to \mathbf{1}$ is in $\mathcal{L}$.

- δ_2 denotes the function induced by the middle-to-right map of spans;
- η_1 denotes the function induced by the middle-to-top map of spans; and
- η_2 denotes the function induced by the middle-to-bottom map of spans.

The 3×3 lemma now gives us an equivalence $\tau_1 : P_H \xrightarrow{\simeq} P_V$ of types defined by double induction on pushouts [11, Section VII].

▶ Note. Let Δ be a discrete graph and G an A-diagram over Δ. The pushout

$$\begin{array}{ccc} \Delta_0 \times A & \xrightarrow{(i,a)\mapsto(i,\mathsf{str}(G_i)(a))} & \sum_{i:\Delta_0} \mathsf{ty}(G_i) \\ \downarrow{\scriptstyle \mathsf{pr}_2} & & \downarrow{\scriptstyle \mathsf{inr}} \\ A & \xrightarrow[\mathsf{inl}]{} & D \end{array}$$

together with inl is the coproduct of the G_i in $A/\mathcal{U}$. We denote D by $\bigvee_{i:\Delta_0} \mathsf{ty}(G_i)$.

▶ **Lemma 19.** *We have two equivalences of spans:*

$$\begin{array}{ccccc} A & \xleftarrow{[\mathsf{id}_A]} & \mathsf{colim}_\Gamma A & \xrightarrow{\psi} & \mathsf{colim}_\Gamma(\mathcal{F}(F)) \\ \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} \\ H_3 & \xleftarrow[\eta_2]{} & H_2 & \xrightarrow[\eta_1]{} & H_1 \end{array}$$

$$\begin{array}{ccccc} V_1 & \xleftarrow{\delta_1} & V_2 & \xrightarrow{\delta_2} & V_3 \\ \| & & \downarrow{\simeq} & & \| \\ \bigvee_{i,j,g} \mathsf{ty}(F_i) & \xleftarrow[\mathsf{id} \vee \mathsf{id}]{} & \left(\bigvee_{i,j,g} \mathsf{ty}(F_i)\right) \vee \left(\bigvee_{i,j,g} \mathsf{ty}(F_i)\right) & \xrightarrow[\nu]{} & \bigvee_i \mathsf{ty}(F_i) \end{array}$$

where ν is defined by double induction on pushouts through the commuting square

$$\begin{array}{ccc} A & \longrightarrow & \bigvee_{i,j,g} \mathsf{pr}_1(F_i) \\ \downarrow & \mathsf{refl}_{\mathsf{inl}(a)} & \downarrow{\scriptstyle (i,j,g,x)\mapsto\mathsf{inr}(j,\mathsf{fun}(F_{i,j,g})(x))} \\ \bigvee_{i,j,g} \mathsf{pr}_1(F_i) & \xrightarrow[(i,j,g,x)\mapsto\mathsf{inr}(i,x)]{} & \bigvee_i \mathsf{pr}_1(F_i) \end{array}.$$

Notice that the pushout of the topmost span appearing in Lemma 19 is exactly $\mathcal{P}_F$. By the 3×3 lemma, this gives us $\mathsf{colim}^A_\Gamma(F)$ as a familiar pushout of coproducts.

▶ **Corollary 20** ([6, Corollary 4.5.3])**.** *We have a pushout square*

$$\begin{array}{ccc} \left(\bigvee_{i,j,g} \mathsf{ty}(F_i)\right) \vee \left(\bigvee_{i,j,g} \mathsf{ty}(F_i)\right) & \xrightarrow{\nu} & \bigvee_i \mathsf{ty}(F_i) \\ \downarrow{\scriptstyle \mathsf{id} \vee \mathsf{id}} & & \downarrow \\ \bigvee_{i,j,g} \mathsf{ty}(F_i) & \longrightarrow & \mathsf{colim}^A_\Gamma(F) \end{array}$$

8.2 Generalized Mayer-Vietoris property

With this new construction of colim^A_Γ, we can transfer the generalized Mayer-Vietoris property of cohomology to HoTT. This application is described in detail in [6, Section 7]. To use the Mayer-Vietoris sequence, we assume the univalence axiom.

Suppose that $H^* : (\mathcal{U}^*)^{\mathrm{op}} \to \mathbf{Ab}$ is a cohomology theory.[8] Consider a pushout

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathsf{inr}} \\ X & \xrightarrow[\mathsf{inl}]{} & P \end{array}$$

of a span of pointed maps. In [4], Cavallo constructs the *Mayer-Vietoris sequence* for P, a long exact sequence (LES) of the form

$$\cdots \quad H^{n-1}(P) \longrightarrow H^{n-1}(X) \times H^{n-1}(Y) \longrightarrow H^{n-1}(Z) \longrightarrow H^n(P) \xrightarrow[(H^n(\mathsf{inl}),H^n(\mathsf{inr}))]{} H^n(X) \times H^n(Y) \xrightarrow[H^n(f)-H^n(g)]{} H^n(Z) \quad \cdots$$

Let F be a diagram of pointed types over a finite graph.[9] As cohomology preserves finite wedges [4, Section 4.2], Corollary 20 combined with this LES gives us an exact squence

$$H^n(\mathsf{colim}^*_\Gamma(F)) \xrightarrow{\zeta_n} \prod_{i,j,g} H^n(F_i) \times \prod_i H^n(F_i) \xrightarrow{\mu_n - \nu_n} \prod_{i,j,g} H^n(F_i) \times \prod_{i,j,g} H^n(F_i) \tag{ext}$$

for each $n : \mathbb{Z}$. Here, ζ_n is defined as the composite

$$\begin{array}{ccc} H^n(\mathsf{colim}^*_\Gamma(F)) & \overset{\zeta_n}{\dashrightarrow} & \prod_{i,j,g} H^n(F_i) \times \prod_i H^n(F_i) \\ \downarrow & \nearrow_{\cong \times \cong} & \\ H^n(\bigvee_{i,j,g} F_i) \times H^n(\bigvee_i F_i) & & \end{array}$$

and μ_n and ν_n are defined by $(f,h) \mapsto (f, \lambda i \lambda j \lambda g. H^n(F_{i,j,g})(h_j))$ and $(f,h) \mapsto (f, \lambda i \lambda j \lambda g. h_i)$, respectively. Moreover, the universal property of limits in $\mathbf{Ab}$ gives us a commuting triangle

$$\begin{array}{ccc} H^n(\mathsf{colim}^*_\Gamma(F)) & \overset{\Delta_F}{\dashrightarrow} & \mathsf{lim}_\Gamma\, H^n(F) \\ & {\scriptstyle H^n(\iota_i)}\searrow \quad \swarrow{\scriptstyle \mathsf{pr}_i} & \\ & H^n(F_i) & \end{array}$$

for each $i : \Gamma_0$, induced by the cone $(H^n(\mathsf{colim}^*_\Gamma(F)), H^n(\iota))$ over $H^n(F)$. One can check that the exactness of (ext) implies that Δ_F is epic as a map of sets.

At this stage, if we were in a classical system, then it would follow that Δ_F has a section, which in turn would imply that $H^n(\mathsf{colim}^*_\Gamma(F))$ is a weak limit. Inside HoTT, we may assume the axiom of choice [23, Section 3.8] to conclude that Δ_F is *merely* a weak limit. In this sense, H^* has the finite generalized Mayer-Vietoris property inside HoTT.

[8] See [1, Section 6] for a description of Eilenberg-Steenrod cohomology theory inside HoTT. A slightly more general definition is found in [6, Section 7].

[9] This means that Γ_0 is finite and $\Gamma_1(i,j)$ is finite for all vertices i and j. When H^* is a singular cohomology theory, we may extend the class of graphs to those satisfying the set-level axiom of choice, in the sense of [1, Definition 6.1].

9 Conclusion and future work

We explored colimits inside HoTT. The heart of our work was the connection between A-colimits and ordinary colimits, i.e., *the main connection*. To derive the main connection, we found an explicit construction of A-colimits that was tailored to reveal the connection. We used the main connection to prove that the forgetful functor from a coslice creates colimits over trees and that A-colimits over trees are universal. We also used the main connection to examine how colimits interact with orthogonal factorization systems. As a result, we found that all pointed colimits preserve n-connectedness, which implies that higher groups are closed under colimits on directed graphs. Finally, we used the main connection to see that cohomology takes finite colimits to weak limits in **Set** assuming the axiom of choice.

Our future work includes additional Agda formalization. In addition, a natural direction is to extend our development to colimits of diagrams over 2-computads [22]. To our knowledge, colimits of type-valued diagrams over higer-dimensional graphs have not been developed in the untruncated setting. We believe both Section 6 and Section 7 can be generalized to the setting of 2-computads.

References

1 Ulrik Buchholtz and Kuen-Bang Hou (Favonia). Cellular Cohomology in Homotopy Type Theory. *Logical Methods in Computer Science*, Volume 16, Issue 2, 2020. doi:10.23638/LMCS-16(2:7)2020.

2 Ulrik Buchholtz, Floris van Doorn, and Egbert Rijke. Higher groups in homotopy type theory. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '18, page 205–214, New York, NY, USA, 2018. Association for Computing Machinery. doi:10.1145/3209108.3209150.

3 Paolo Capriotti and Nicolai Kraus. Univalent higher categories via complete Semi-Segal types. *Proc. ACM Program. Lang.*, 2(POPL), 2017. doi:10.1145/3158132.

4 Evan Cavallo. Synthetic cohomology in homotopy type theory, 2015. URL: https://staff.math.su.se/evan.cavallo/works/thesis15.pdf.

5 J Daniel Christensen and Luis Scoccola. The Hurewicz theorem in homotopy type theory. *Algebraic and Geometric Topology*, 23(5):2107–2140, 2023. doi:10.2140/agt.2023.23.2107.

6 Perry Hart and Kuen-Bang Hou (Favonia). Technical report for "colimits in homotopy type theory", 2024. URL: https://phart3.github.io/colimits-paper-TR.pdf.

7 Kuen-Bang Hou (Favonia), Eric Finster, Daniel R. Licata, and Peter LeFanu Lumsdaine. A Mechanization of the Blakers-Massey Connectivity Theorem in Homotopy Type Theory. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '16, page 565–574, New York, NY, USA, 2016. Association for Computing Machinery. doi:10.1145/2933575.2934545.

8 Krzysztof Kapulkin and Peter LeFanu Lumsdaine. The simplicial model of Univalent Foundations (after Voevodsky). *J. Eur. Math. Soc.*, 23(6):2071–2126, 2021. doi:10.4171/JEMS/1050.

9 Max Kelly. On maclane's conditions for coherence of natural associativities, commutativities, etc. *Journal of Algebra*, 1(4):397–402, 1964. doi:10.1016/0021-8693(64)90018-3.

10 Nicolai Kraus and Jakob von Raumer. Path spaces of higher inductive types in homotopy type theory. In *Proceedings of the 34th Annual ACM/IEEE Symposium on Logic in Computer Science*, 2021.

11 Daniel R. Licata and Guillaume Brunerie. A cubical approach to synthetic homotopy theory. In *2015 30th Annual ACM/IEEE Symposium on Logic in Computer Science*. IEEE, 2015. doi:10.1109/LICS.2015.19.

12 Jacob Lurie. Higher algebra. Unpublished. Available online at https://www.math.ias.edu/~lurie/, 2017.

13 nLab authors. created limit. https://ncatlab.org/nlab/show/created+limit, 2024. Revision 21.

14 nLab authors. (infinity,1)-limit. https://ncatlab.org/nlab/show/%28%E2%88%9E%2C1%29-limit, 2024. Revision 78.

15 Emily Riehl. *Categorical Homotopy Theory*. New Mathematical Monographs. Cambridge University Press, 2014. doi:10.1017/CBO9781107261457.

16 Egbert Rijke. Classifying types, 2019. URL: https://arxiv.org/abs/1906.09435.

17 Egbert Rijke. Introduction to homotopy type theory, 2022. URL: https://arxiv.org/abs/2212.11082, arXiv:2212.11082.

18 Egbert Rijke, Michael Shulman, and Bas Spitters. Modalities in homotopy type theory. *Logical Methods in Computer Science*, Volume 16, Issue 1, 2020.

19 Egbert Rijke, Elisabeth Stenholm, Jonathan Prieto-Cubides, Fredrik Bakke, and others. The agda-unimath library. URL: https://github.com/UniMath/agda-unimath/.

20 Kristina Sojakova. Higher inductive types as homotopy-initial algebras. *SIGPLAN Not.*, 50(1):31–42, 2015. doi:10.1145/2775051.2676983.

21 Kristina Sojakova, Floris van Doorn, and Egbert Rijke. Sequential colimits in homotopy type theory. In *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '20, page 845–858, 2020. doi:10.1145/3373718.3394801.

22 Ross Street. Limits indexed by category-valued 2-functors. *Journal of Pure and Applied Algebra*, 8(2):149–181, 1976. doi:10.1016/0022-4049(76)90013-X.

23 The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. https://homotopytypetheory.org/book, 2013.

24 Floris van Doorn, Jakob von Raumer, and Ulrik Buchholtz. Homotopy type theory in lean. In *Interactive Theorem Proving*, pages 479–495. Springer International Publishing, 2017. doi:10.1007/978-3-319-66107-0_30.