Coslice Colimits in Homotopy Type Theory

Perry Hart Kuen-Bang Hou (Favonia)

University of Minnesota, Twin Cities

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Abstract

We contribute to the theory of (homotopy) colimits inside homotopy type theory. The heart of our work characterizes the connection between colimits in coslices of a universe, called *coslice colimits*, and colimits in the universe (i.e., ordinary colimits). To derive this characterization, we find an explicit construction of colimits in coslices that is tailored to reveal the connection. We use the construction to derive properties of colimits. Notably, we prove that the forgetful functor from a coslice creates colimits over trees. We also use the construction to examine how colimits interact with orthogonal factorization systems and with cohomology theories. As a consequence of their interaction with orthogonal factorization systems, all pointed colimits (special kinds of coslice colimits) preserve n-connectedness, which implies that higher groups are closed under colimits on directed graphs. We have formalized our main construction of the coslice colimit functor in Agda. The code for this paper is available at `https://github.com/PHart3/colimits-agda`.

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1 Introduction

Working in homotopy type theory (HoTT), we study higher inductive types (HITs) arising as *(homotopy) colimits* in coslices of a universe, called *coslice colimits*. Coslices of a universe are type-theoretic versions of coslice categories. Our study of coslice colimits is organized as follows.

The main connection (Section 5.4)

Suppose $\mathcal{U}$ is a universe and A is a type in $\mathcal{U}$. We want to construct all colimits in $A/\mathcal{U}$, or A-*colimits*. The (wild) category $A/\mathcal{U}$ has objects $\sum_{T:\mathcal{U}} A \to T$ with $X \to_A Y :\equiv \sum_{k:\mathsf{pr}_1(X)\to\mathsf{pr}_1(Y)} k \circ \mathsf{pr}_2(X) \sim \mathsf{pr}_2(Y)$ as morphisms from X to Y. HoTT has a general schema for HITs that would let us simply postulate A-colimits. We, however, explicitly construct A-colimits with just the machinery of Martin-Löf type theory (MLTT) augmented with pushouts. We take this different approach to reveal the connection between A-colimits and their underlying colimits in $\mathcal{U}$. Therefore, we call the construction of Section 5.4 the *main connection*. In fact, our construction is *not* a case of a general method to encode higher-dimensional HITs with pushouts but rather tailored to reveal this connection.

This connection sheds light on other areas of synthetic homotopy theory, which we discuss now.

The universality of colimits (Section 6)

The *universality* of colimits is the defining feature of locally cartesian closed (LCC) ∞-categories, such as that of spaces. The main connection will establish a well-known classical result inside type theory: The forgetful functor $A/\mathcal{U} \to \mathcal{U}$ *creates* colimits of diagrams over contractible graphs (Corollary 5.4.6). We review such graphs, known as *trees*, in Section 4. With the forgetful functor creating colimits, we can transfer universality of ordinary colimits to A-colimits in many cases (Corollary 6.0.3). This is notable as LCC ∞-categories are not closed under coslices.

The categories of higher groups are cocomplete (Section 7)

A striking feature of colimits is their interaction with (orthogonal) factorization systems. In Section 7, we use the main connection to show that colimits in $A/\mathcal{U}$ preserve left classes of maps of such systems on $\mathcal{U}$. It is significant that we consider systems on $\mathcal{U}$ rather than $A/\mathcal{U}$. We could derive a similar preservation theorem for systems on $A/\mathcal{U}$ directly from the universal property of an A-colimit. In practice, however, the factorization systems we tend to care about are on $\mathcal{U}$. Since the main connection relates the action of A-colimits on maps to the action of their underlying colimits on maps, we manage to deduce the preservation theorem for systems on $\mathcal{U}$.

To prove this theorem, we find it useful to develop the theory of factorization systems in a more general setting than $\mathcal{U}$. In Section 3.3, we study such systems on *wild categories*, which make up one approach to category theory in HoTT. We prove that if a functor F of well-behaved wild categories with factorization systems has a right adjoint G, then F preserves the left class when G preserves the right class (Corollary 3.3.9). We combine this result with the main connection to deduce the desired preservation property.

When we focus on the (n-connected, n-truncated) system on $\mathcal{U}$ [14, Chapter 7.6] and take A as the unit type, the main connection shows that the colimit of every diagram of pointed n-connected types is n-connected. One useful corollary of this is that the higher category $(n, k)\,\mathsf{GType}$ of k-tuply groupal n-groupoids considered by [5] is cocomplete on (directed) graphs for all truncation levels $-2 \leq n \leq \infty$ and $-1 \leq k < \infty$ (Section 7.1). We also exploit the generality of the main connection to extend this cocompleteness result to categories of *higher pointed abelian groups* (Corollary 7.1.3).

Cohomology sends colimits to weak limits (Section 8)

Finally, we examine how colimits interact with cohomology theories, which are important algebraic invariants of spaces. To do so, we consider *weak limits*, which are key ingredients in the Brown representability theorem. A weak colimit in a category need not satify the uniqueness property required of a colimit. The Brown representability theorem specifies conditions for a presheaf on the homotopy category $\mathbf{Ho}(\mathbf{Top}_{*,c})$ of pointed connected spaces to be representable. The known proof of this theorem requires the presheaf to send countable homotopy colimits in $\mathbf{Top}_{*,c}$ to weak limits in **Set**. Eilenberg-Steenrod cohomology theories enjoy this property as set-valued functors.

In Section 5.5, we use the main connection to establish a restricted, type-theoretic version of this property. From the main connection we derive another construction of A-colimits, as pushouts of coproducts (Corollary 5.5.3), which mirrors a well-known classical lemma. We take A as the unit type and combine the new construction with the Mayer-Vietoris sequence to find that cohomology takes finite colimits to weak limits assuming the axiom of choice.

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2 Type system

We assume the reader is familiar with MLTT and HITs in the style of [14]. We will work in MLTT augmented with ordinary colimits and denote this system by $\mathsf{MLTT} + \mathsf{Colim}$. In particular, the entirety of Section 5.4 takes place inside $\mathsf{MLTT} + \mathsf{Colim}$. In fact, we need only augment MLTT with pushouts as they let us construct all nonrecursive 1-HITs, including ordinary colimits, with all of their computational properties. Notably, $\mathsf{MLTT} + \mathsf{Colim}$ comes with strong function extensionality for free. This property is critical for reasoning about functions in type theory and underlies almost all our work.

Remarks on notation

- The symbol $=$ denotes the identity type. The symbol $\equiv$ denotes definitional equality. The symbol $:\equiv$ denotes term definition.
- Either underbrace or overbrace may be used to indicate $\equiv$, $:\equiv$, or $=$.
- We may use the Agda notation $(x : X) \to Y(x)$ for the type $\prod_{x:X} Y(x)$.
- For convenience, we'll use the notation

$$\mathsf{PI}(p_1, \ldots, p_n) \ : \ a = b$$

to denote an equality obtained by simultaneous or iterative path induction on paths $p_1, \ldots, p_n$. We only use this shorthand when the equality is constructed in an evident way.

3 Categorical notions

3.1 Wild categories and functors

Definition 3.1.1. Let $\mathcal{U}$ be a universe. A *wild category (in $\mathcal{U}$)* is a tuple consisting of

$$\begin{aligned}
\mathsf{Ob} &: \mathcal{U} \\
\mathsf{hom} &: \mathsf{Ob} \to \mathsf{Ob} \to \mathcal{U} \\
\circ &: \prod_{X,Y,Z:\mathsf{Ob}} \mathsf{hom}(Y,Z) \to \mathsf{hom}(X,Y) \to \mathsf{hom}(X,Z) \\
\mathsf{id} &: \prod_{X:\mathsf{Ob}} \mathsf{hom}(X,X) \\
\mathsf{RId} &: \prod_{X,Y:\mathsf{Ob}} \prod_{f:\mathsf{hom}(X,Y)} f \circ \mathsf{id}_X = f \\
\mathsf{LId} &: \prod_{X,Y:\mathsf{Ob}} \prod_{f:\mathsf{hom}(X,Y)} \mathsf{id}_Y \circ f = f \\
\mathsf{assoc} &: \prod_{W,X,Y,Z:\mathsf{Ob}} \prod_{h:\mathsf{hom}(Y,Z)} \prod_{g:\mathsf{hom}(X,Y)} \prod_{f:\mathsf{hom}(W,X)} (h \circ g) \circ f = h \circ (g \circ f)
\end{aligned}$$

Definition 3.1.2. A *bicategory* is a wild category $\mathcal{C}$ equipped with

- an identity

$$\begin{array}{c}
\mathsf{ap}_{-\circ f}(\mathsf{assoc}(k,g,h)) \cdot \mathsf{assoc}(k, g \circ h, f) \cdot \mathsf{ap}_{k\circ -}(\mathsf{assoc}(g,h,k)) \\
\Big\| {\scriptstyle \rho(f,k,g,h)} \\
\mathsf{assoc}(k \circ g, h, f) \cdot \mathsf{assoc}(k, g, h \circ f)
\end{array}$$

for all composable morphism k, g, h, and f.

- an identity

$$\begin{array}{c}\mathsf{assoc}(g,\mathsf{id},h)\cdot \mathsf{ap}_{g\circ -}(\mathsf{Lid}(h)) \\ \Big\Vert{\scriptstyle v(g,h)} \\ \mathsf{ap}_{-\circ h}(\mathsf{Rid}(g))\end{array}$$

  for all composable morphisms g and h.

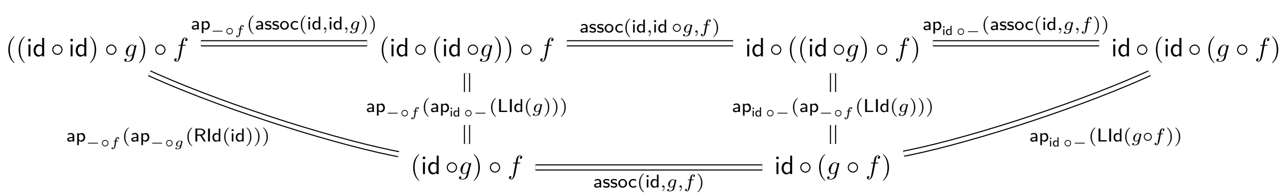
Lemma 3.1.3. *Let $\mathcal{C}$ be a bicategory. For every $A, B, C : \mathsf{Ob}(\mathcal{C})$, $f : \mathsf{hom}_{\mathcal{C}}(A, B)$, $g : \mathsf{hom}_{\mathcal{C}}(B, C)$, we have that*

$$\mathsf{Lid}(g\circ f)^{-1}\cdot \mathsf{assoc}(\mathsf{id},g,f)^{-1}\cdot \mathsf{ap}_{-\circ f}(\mathsf{Lid}(g)) \;=\; \mathsf{refl}_{g\circ f}.$$

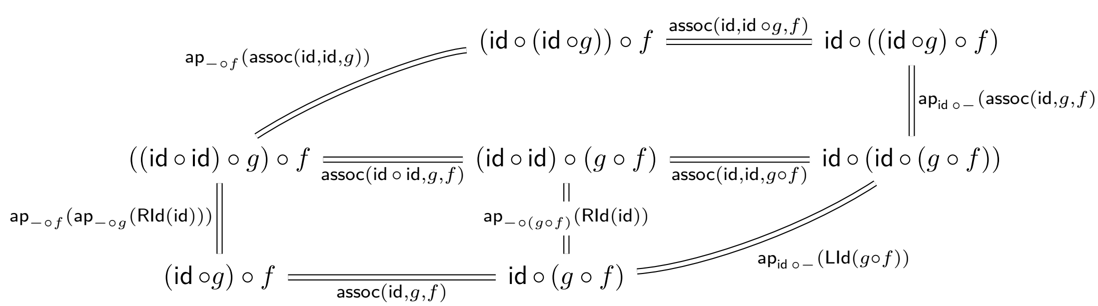
Proof. Since the function $(c = d) \xrightarrow{\mathsf{ap}_{\mathsf{id}\circ -}} (\mathsf{id}\circ c = \mathsf{id}\circ d)$ has a retraction for all parallel morphisms c and d, it suffices to prove that

$$\mathsf{ap}_{\mathsf{id}\circ -}(\mathsf{Lid}(g\circ f))^{-1}\cdot \mathsf{ap}_{\mathsf{id}\circ -}(\mathsf{assoc}(\mathsf{id},g,f))^{-1}\cdot \mathsf{ap}_{\mathsf{id}\circ -}(\mathsf{ap}_{-\circ f}(\mathsf{Lid}(g))) \;=\; \mathsf{refl}_{\mathsf{id}\circ(g\circ f)}.$$

Note that the left two subdiagrams of



commute, and we want to prove that the right one commutes. Hence it suffices to prove that this diagram's outer perimeter commutes. This follows from the commuting diagram



□

Definition 3.1.4. Let $\mathcal{C}$ be a wild category.

- A morphism $f : \mathsf{hom}_{\mathcal{C}}(A, B)$ of $\mathcal{C}$ is an *equivalence* if it is biinvertible:

$$\mathsf{is\_equiv}(f) \;:=\; \sum_{g,h:\mathsf{hom}_{\mathcal{C}}(B,A)} g\circ f = \mathsf{id}_A \times f\circ h = \mathsf{id}_B$$

We write $A \simeq_{\mathcal{C}} B$ for the type of equivalences from A to B.

- We say that $\mathcal{C}$ is *univalent* if the function

$$\begin{aligned} \mathsf{idtoequiv} &: \bigl(A =_{\mathsf{Ob}(\mathcal{C})} B\bigr) \to (A \simeq_{\mathcal{C}} B) \\ \mathsf{refl}_A &\mapsto (\mathsf{id}_A, \mathsf{id}_A, \mathsf{id}_A, \mathsf{LId}(\mathsf{id}_A), \mathsf{LId}(\mathsf{id}_A)) \end{aligned}$$

is an equivalence.

Example 3.1.5. The category $\mathcal{U}$ of types and functions is a bicategory and (assuming the univalence axiom) is univalent.

Definition 3.1.6. A functor $F : \mathcal{C} \to \mathcal{D}$ between wild categories is a tuple consisting of

$$\begin{aligned} F_0 &: \mathsf{Ob}(\mathcal{C}) \to \mathsf{Ob}(\mathcal{D}) \\ F_1 &: \prod_{X,Y:\mathsf{Ob}(\mathcal{C})} \mathsf{hom}_{\mathcal{C}}(X,Y) \to \mathsf{hom}_{\mathcal{D}}(F_0(X), F_0(Y)) \\ F_2 &: \prod_{X,Y,Z:\mathsf{Ob}(\mathcal{C})} \prod_{g:\mathsf{hom}_{\mathcal{C}}(Y,Z)} \prod_{f:\mathsf{hom}_{\mathcal{C}}(X,Y)} F_1(g \circ f) = F_1(g) \circ F_1(f) \\ F_3 &: \prod_{X:\mathsf{Ob}(\mathcal{C})} F_1(\mathsf{id}_X) = \mathsf{id}_{F_0(X)} \end{aligned}$$

We may refer to F_0 or F_1 by F. Also, if the data F_2 and F_3 are omitted, then we call F a 0-*functor*.

Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be 0-functors of wild categories.

Definition 3.1.7. An adjunction $L \dashv R$ consists of terms

$$\begin{aligned} \alpha &: \prod_{A:\mathsf{Ob}(\mathcal{C})} \prod_{X:\mathsf{Ob}(\mathcal{D})} \mathsf{hom}_{\mathcal{D}}(LA, X) \simeq \mathsf{hom}_{\mathcal{C}}(A, RX) \\ V_1 &: \prod_{A:\mathsf{Ob}(\mathcal{C})} \prod_{X,Y:\mathsf{Ob}(\mathcal{D})} \prod_{g:\mathsf{hom}_{\mathcal{D}}(X,Y)} \prod_{h:\mathsf{hom}_{\mathcal{D}}(LA,X)} Rg \circ \alpha(h) = \alpha(g \circ h) \\ V_2 &: \prod_{Y:\mathsf{Ob}(\mathcal{D})} \prod_{A,B:\mathsf{Ob}(\mathcal{C})} \prod_{f:\mathsf{hom}_{\mathcal{C}}(A,B)} \prod_{h:\mathsf{hom}_{\mathcal{D}}(LB,Y)} \alpha(h) \circ f = \alpha(h \circ Lf). \end{aligned}$$

Note that for each such triple, we also have naturality squares

$$\begin{array}{ccc} \mathsf{hom}_{\mathcal{C}}(A, RX) & \xrightarrow{Rg\circ -} & \mathsf{hom}_{\mathcal{C}}(A, RY) \\ {\scriptstyle \alpha^{-1}}\downarrow & \widetilde{V_1}(g) & \downarrow{\scriptstyle \alpha^{-1}} \\ \mathsf{hom}_{\mathcal{D}}(LA, X) & \xrightarrow[g\circ -]{} & \mathsf{hom}_{\mathcal{D}}(LA, Y) \end{array}$$

$$\begin{array}{ccc} \mathsf{hom}_{\mathcal{C}}(B, RY) & \xrightarrow{-\circ f} & \mathsf{hom}_{\mathcal{C}}(A, RY) \\ {\scriptstyle \alpha^{-1}}\downarrow & \widetilde{V_2}(f) & \downarrow{\scriptstyle \alpha^{-1}} \\ \mathsf{hom}_{\mathcal{D}}(LB, Y) & \xrightarrow[-\circ Lf]{} & \mathsf{hom}_{\mathcal{D}}(LA, Y) \end{array}$$

Here, the homotopies witnessing these square commute are defined by

$$\begin{array}{ccc}
g \circ \alpha^{-1}(h) & \overset{\widetilde{V}_1(g,h)}{=\!=\!=\!=\!=\!=} & \alpha^{-1}(Rg \circ h) \\
{\scriptstyle \eta_\alpha(g \circ \alpha^{-1}(h))^{-1}} \Big\Downarrow & & \Big\Uparrow {\scriptstyle \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{Rg\circ -}(\epsilon_\alpha(h)))} \\
\alpha^{-1}(\alpha(g \circ \alpha^{-1}(h))) & \underset{\mathsf{ap}_{\alpha^{-1}}(V_1(g,\alpha^{-1}(h)))^{-1}}{\Longrightarrow} & \alpha^{-1}(Rg \circ \alpha(\alpha^{-1}(h)))
\end{array}$$

$$\begin{array}{ccc}
\alpha^{-1}(h) \circ Lf & \overset{\widetilde{V}_2(f,h)}{=\!=\!=\!=\!=\!=} & \alpha^{-1}(h \circ f) \\
{\scriptstyle \eta_\alpha(\alpha^{-1}(h) \circ Lf)^{-1}} \Big\Downarrow & & \Big\Uparrow {\scriptstyle \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{-\circ f}(\epsilon_\alpha(h)))} \\
\alpha^{-1}(\alpha(\alpha^{-1}(h) \circ Lf)) & \underset{\mathsf{ap}_{\alpha^{-1}}(V_2(f,\alpha^{-1}(h)))^{-1}}{\Longrightarrow} & \alpha^{-1}\big(\big(\alpha(\alpha^{-1}(h))\big) \circ f\big)
\end{array}$$

where η_α and ϵ_α come from the half-adjoint equivalence data of α.

Note 3.1.8. For all $f : \mathsf{hom}_{\mathcal{C}}(A, B)$ and $v : \mathsf{hom}_{\mathcal{C}}(LB, Y)$, we have a chain $\mathsf{exch}(f, v)$ of paths:

$$\begin{array}{c}
\widetilde{V}_2(f, \alpha(v)) \\
\Big\| \ \textit{definitional} \\
\eta_\alpha(\alpha^{-1}(\alpha(v)) \circ Lf)^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(V_2(f, \alpha^{-1}(\alpha(v))))^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{-\circ f}(\epsilon_\alpha(\alpha(v)))) \\
\Big\| \ \textit{via homotopy naturality of } V_2(f,-) \\
\eta_\alpha(\alpha^{-1}(\alpha(v)) \circ Lf)^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{-\circ f}(\mathsf{ap}_\alpha(\eta_\alpha(v))) \cdot V_2(f, v) \cdot \mathsf{ap}_\alpha(\mathsf{ap}_{-\circ Lf}(\eta_\alpha(v)))^{-1})^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{-\circ f}(\epsilon_\alpha(\alpha(v)))) \\
\Big\| \\
\eta_\alpha(\alpha^{-1}(\alpha(v)) \circ Lf)^{-1} \cdot \mathsf{ap}_{\alpha^{-1}\circ\alpha}(\mathsf{ap}_{-\circ Lf}(\eta_\alpha(v))) \cdot \mathsf{ap}_{\alpha^{-1}}(V_2(f, v))^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{-\circ f}(\mathsf{ap}_\alpha(\eta_\alpha(v))))^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{-\circ f}(\epsilon_\alpha(\alpha(v)))) \\
\Big\| \ \textit{via homotopy naturality of } \eta_\alpha \\
\mathsf{ap}_{-\circ Lf}(\eta_\alpha(v)) \cdot \eta_\alpha(v \circ Lf)^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(V_2(f, v))^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{-\circ f}(\mathsf{ap}_\alpha(\eta_\alpha(v))))^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{-\circ f}(\epsilon_\alpha(\alpha(v)))) \\
\Big\| \ \textit{via half-adjoint equivalence condition} \\
\mathsf{ap}_{-\circ Lf}(\eta_\alpha(v)) \cdot \eta_\alpha(v \circ Lf)^{-1} \cdot \mathsf{ap}_{\alpha^{-1}}(V_2(f, v))^{-1}
\end{array}$$

3.2 Reflective subcategories

Definition 3.2.1. Let $\mathcal{C}$ be a wild category. A *reflective subcategory* of $\mathcal{C}$ is a predicate $P : \mathsf{Ob}(\mathcal{C}) \to \mathsf{Prop}$ together with functions

$$\bigcirc \ : \ \mathsf{Ob}(\mathcal{C}) \to \mathsf{Ob}(\mathcal{C}) \qquad \eta \ : \ \prod_{X : \mathsf{Ob}(\mathcal{C})} \mathsf{hom}_{\mathcal{C}}(X, \bigcirc X)$$

such that

- for each $X : \mathsf{Ob}(\mathcal{C})$, $P(\bigcirc X)$
- for each $X, Y : \mathsf{Ob}(\mathcal{C})$ with $P(Y)$, the function $(- \circ \eta_X) : \mathsf{hom}_{\mathcal{C}}(\bigcirc X, Y) \to \mathsf{hom}_{\mathcal{C}}(X, Y)$ is an equivalence. The inverse of this map is denoted by $\mathsf{rec}_{\bigcirc}$, which has a homotopy

$\beta_\eta : \prod_{X,Y:\mathsf{Ob}(\mathcal{C})} \prod_{\_:P(Y)} \prod_{f:\mathsf{hom}_{\mathcal{C}}(X,Y)} \mathsf{rec}_{\bigcirc}(f) \circ \eta_X = f$.[1]

We define $\mathcal{C}_P :\equiv \sum_{X:\mathsf{Ob}(\mathcal{C})} P(X)$.

Suppose that $\mathcal{C}$ is a wild category. Let $(P, \bigcirc, \eta)$ be a reflective subcategory of $\mathcal{C}$.

Proposition 3.2.2. *For each $X : \mathsf{Ob}(\mathcal{C})$, the following square commutes in $\mathcal{C}$:*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle \eta_X}\big\downarrow & & \big\downarrow{\scriptstyle \eta_Y} \\ \bigcirc X & \xrightarrow[\mathsf{rec}_{\bigcirc}(\eta_Y \circ f)]{} & \bigcirc Y \end{array}$$

Lemma 3.2.3. *Suppose that $\mathcal{C}$ is univalent. For each $X : \mathsf{Ob}(\mathcal{C})$, $P(X) \to \mathsf{is\_equiv}(\eta_X)$.*

Proof. Let $X : \mathsf{Ob}(\mathcal{C})$. The type $T_{P,X}$ of tuples

$$\begin{aligned} Y &: \mathsf{Ob}(\mathcal{C}) \\ q &: P(Y) \\ f &: \mathsf{hom}_{\mathcal{C}}(X,Y) \\ I &: \prod_{Z:\mathsf{Ob}(\mathcal{C})} P(Z) \to \mathsf{is\_equiv}(\lambda(g : \mathsf{hom}_{\mathcal{C}}(Y,Z)).g \circ f) \end{aligned}$$

is a mere proposition. Suppose that $P(X)$. We have terms

$$(X, \ldots, \mathsf{id}_X, \ldots) \qquad (\bigcirc X, \ldots, \eta_X, \ldots)$$

of type $T_{P,X}$, which must be equal. Therefore, we have a commuting triangle

$$\begin{array}{ccccc} & & X & & \\ & {\scriptstyle \mathsf{id}}\swarrow & & \searrow{\scriptstyle \eta_X} & \\ X & & \overset{\simeq}{\dashrightarrow} & & \bigcirc X \end{array}$$

in $\mathcal{C}$. This implies that η_X is an equivalence. ◻

Combined with Proposition 3.2.2, Lemma 3.2.3 implies that when $\mathcal{C}$ is univalent, η restricted to $\mathcal{C}_P$ is a *natural* isomorphsm $\mathsf{id}_{\mathcal{C}_P} \xrightarrow{\simeq} \bigcirc \circ \mathcal{I}$ of functors where $\mathcal{I}$ denotes the inclusion of the subuniverse $\mathcal{C}_P$ into $\mathcal{C}$. This observation motivates the following definition (see Definition B.0.1 for the notion of *2-coherent* left adjoint).

Definition 3.2.4. Let $\mathcal{C}$ and $\mathcal{D}$ be wild categories. We say that $\mathcal{D}$ is *reflective in* $\mathcal{C}$ if we have a 2-coherent left adjoint $L : \mathcal{C} \to \mathcal{D}$ whose counit is a natural isomorphism of 0-functors.

[1]When $\mathcal{C} \equiv \mathcal{U}$, a reflective subcategory is known as a *reflective subuniverse*.

3.3 Orthogonal factorization systems

Definition 3.3.1. Let $\mathcal{C}$ be a wild category. An *orthogonal factorization system (OFS)* on $\mathcal{C}$ consists of predicates $\mathcal{L}, \mathcal{R} \; : \; \prod_{A,B \,:\, \mathcal{C}} \mathsf{hom}_{\mathcal{C}}(A,B) \to \mathsf{Prop}$ such that

1. both $\mathcal{L}$ and $\mathcal{R}$ are closed under composition and have all identities;

2. for every $h : \mathsf{hom}_{\mathcal{C}}(A,B)$, the following type is contractible:

$$\mathsf{fact}_{\mathcal{L},\mathcal{R}}(h) \;\coloneqq\; \sum_{D:\mathsf{Ob}(\mathcal{C})} \; \sum_{f:\mathsf{hom}_{\mathcal{C}}(A,D)} \; \sum_{g:\mathsf{hom}_{\mathcal{C}}(D,B)} (g \circ f) = h \times \mathcal{L}(f) \times \mathcal{R}(g)$$

Example 3.3.2. Rijke et al. use a particular indexed recursive 1-HIT to show that every family $\prod_{a:A} F(a) \to G(a)$ of functions induces an OFS on $\mathcal{U}$ [11, Section 2.4].

Definition 3.3.3 (Lifting property). Let $\mathcal{C}$ be a wild category. Let $l : \mathsf{hom}_{\mathcal{C}}(A,B)$ and $\mathcal{H}$ be a property of morphisms in $\mathcal{C}$. We say that l has the *left lifting property against* $\mathcal{H}$ if for every $r : \mathsf{hom}_{\mathcal{C}}(C,D)$ with $r \in \mathcal{H}$ and every commuting square

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle l}\big\downarrow & S & \big\downarrow{\scriptstyle r} \\ B & \xrightarrow[g]{} & D \end{array}$$

the type of diagonal fillers

$$\mathsf{fill}(S) \;\coloneqq\; \sum_{d:\mathsf{hom}_{\mathcal{C}}(B,C)} \; \sum_{H_f : d \circ l = f} \; \sum_{H_g : r \circ d = g} \mathsf{assoc}(r,d,l) \cdot \mathsf{ap}_{r \circ -}(H_f) = \mathsf{ap}_{- \circ l}(H_g) \cdot S$$

is contractible. In this case, we write ${}^{\perp}\mathcal{H}(l)$. The predicate *right lifting property* is defined similarly.

Let $\mathcal{C}$ be a univalent bicategory and $(\mathcal{L},\mathcal{R})$ be an OFS on $\mathcal{C}$.

Lemma 3.3.4. *Let $h : \mathsf{hom}_{\mathcal{C}}(N,M)$ and $(U, s_U, t_U, p_U), (V, s_V, t_V, p_V) : \mathsf{fact}_{\mathcal{L},\mathcal{R}}(h)$. We have that*

$$\begin{gathered} (U, s_U, t_U, p_U) = (V, s_V, t_V, p_V) \\ \wr\! \mid \\ \textstyle\sum_{e:U \simeq_{\mathcal{C}} V} \underbrace{\sum_{H_{\mathcal{L}} : s_V = e \circ s_U} \; \sum_{H_{\mathcal{R}} : t_U = t_V \circ e} \mathsf{ap}_{t_V \circ -}(H_{\mathcal{L}}) \cdot \mathsf{assoc}(t_V, e, s_U)^{-1} \cdot \mathsf{ap}_{- \circ s_U}(H_{\mathcal{R}})^{-1} \cdot p_U = p_V}_{\mathfrak{A}(V, s_V, t_V, p_V, e)} \end{gathered}$$

Proof. For each $U : \mathsf{Ob}(\mathcal{C})$, the type family $V \mapsto U \simeq_{\mathcal{C}} V$ is an identity system on $(\mathsf{Ob}(\mathcal{C}), U)$ because

$\mathcal{C}$ is univalent. Moreover,

$$\begin{gathered}
\sum_{s_V:\mathsf{hom}(N,U)}\sum_{t_V:\mathsf{hom}(U,M)}\sum_{p_V:t_V\circ s_V=h}\sum_{H_\mathcal{L}:s_V=\mathsf{id}_U\circ s_U}\\
\sum_{H_\mathcal{R}:t_U=t_V\circ\mathsf{id}_U}\mathsf{ap}_{t_V\circ -}(H_\mathcal{L})\cdot\mathsf{assoc}(t_V,\mathsf{id}_U,s_U)^{-1}\cdot\mathsf{ap}_{-\circ s_U}(H_\mathcal{R})^{-1}\cdot p_U=p_V\\
\wr|\\
\sum_{s_V:\mathsf{hom}(N,U)}\sum_{H_\mathcal{L}:s_V=\mathsf{id}_U\circ s_U}\sum_{t_V:\mathsf{hom}(U,M)}\sum_{H_\mathcal{R}:t_U=t_V\circ\mathsf{id}_U}\\
\sum_{p_V:t_V\circ s_V=h}\mathsf{ap}_{t_V\circ -}(H_\mathcal{L})\cdot\mathsf{assoc}(t_V,\mathsf{id}_U,s_U)^{-1}\cdot\mathsf{ap}_{-\circ s_U}(H_\mathcal{R})^{-1}\cdot p_U=p_V\\
\wr|\\
\sum_{t_V:\mathsf{hom}(U,M)}\sum_{H_\mathcal{R}:t_U=t_V\circ\mathsf{id}_U}\sum_{p_V:t_V\circ(\mathsf{id}_U\circ s_U)=h}\mathsf{assoc}(t_V,\mathsf{id}_U,s_U)^{-1}\cdot\mathsf{ap}_{-\circ s_U}(H_\mathcal{R})^{-1}\cdot p_U=p_V\\
\wr|\\
\sum_{t_V:\mathsf{hom}(U,M)}\sum_{H_\mathcal{R}:t_U=t_V}\sum_{p_V:t_V\circ(\mathsf{id}_U\circ s_U)=h}\mathsf{assoc}(t_V,\mathsf{id}_U,s_U)^{-1}\cdot\mathsf{ap}_{-\circ s_U}(H_\mathcal{R}\cdot\mathsf{RId}(t_V)^{-1})^{-1}\cdot p_U=p_V\\
\wr|\\
\sum_{p_V:t_U\circ(\mathsf{id}_U\circ s_U)=h}\mathsf{assoc}(t_U,\mathsf{id}_U,s_U)^{-1}\cdot\mathsf{ap}_{-\circ s_U}(\mathsf{RId}(t_U)^{-1})^{-1}\cdot p_U=p_V\\
\wr|\\
\mathbf{1}
\end{gathered}$$

Therefore, by Theorem A.0.3, we have our desired equivalence. ☐

Lemma 3.3.5 (Unique lifting property). *For each commuting square*

$$\begin{array}{ccc}
A & \xrightarrow{f} & X\\
{\scriptstyle l}\downarrow & S & \downarrow{\scriptstyle r}\\
B & \xrightarrow[g]{} & Y
\end{array}$$

such that $l\in\mathcal{L}$ and $r\in\mathcal{R}$, the type $\mathsf{fill}(S)$ *of diagonal fillers is contractible.*

Proof. We have a commuting diagram

$$\begin{array}{ccccc}
 & & \overset{f}{\frown}\ {\scriptstyle p_f} & & \\
A & \xrightarrow{s_f} & \mathrm{im}(f) & \xrightarrow{t_f} & X\\
{\scriptstyle l}\downarrow & & & & \downarrow{\scriptstyle r}\\
B & \xrightarrow{s_g} & \mathrm{im}(g) & \xrightarrow{t_g} & Y\\
 & & \underset{g}{\smile}\ {\scriptstyle p_g} & &
\end{array}$$

Since $\mathsf{fact}_{\mathcal{L},\mathcal{R}}(r\circ f)$ is contractible, so is its identity type

$$\big(\mathrm{im}(f),s_f,r\circ t_f,\mathsf{assoc}(r,t_f,s_f)\cdot\mathsf{ap}_{r\circ -}(p_f)\big)\ =\ \big(\mathrm{im}(g),s_g\circ l,t_g,\mathsf{assoc}(t_g,s_g,l)^{-1}\cdot\mathsf{ap}_{-\circ l}(p_g)\cdot S\big)$$

By Lemma 3.3.4, the following type is also contractible:

$$\begin{gathered}
\sum_{e:\mathrm{im}(f)\simeq_\mathcal{C}\mathrm{im}(g)}\sum_{H_\mathcal{L}:s_g\circ l=e\circ s_f}\\
\sum_{H_\mathcal{R}:r\circ t_f=t_g\circ e}\mathsf{ap}_{t_g\circ -}(H_\mathcal{L})\cdot\mathsf{assoc}(t_g,e,s_f)^{-1}\cdot\mathsf{ap}_{-\circ s_f}(H_\mathcal{R})^{-1}\cdot\mathsf{assoc}(r,t_f,s_f)\cdot\mathsf{ap}_{r\circ -}(p_f)=\mathsf{assoc}(t_g,s_g,l)^{-1}\cdot\mathsf{ap}_{-\circ l}(p_g)\cdot S
\end{gathered}$$

By the bicategory structure of $\mathcal{C}$, we have the chain of equivalences shown on the next page.

$$\sum_{e:\mathrm{im}(f)\simeq_{\mathcal{C}}\mathrm{im}(g)} \sum_{H_{\mathcal{L}}:s_g\circ l=e\circ s_f} \sum_{H_{\mathcal{R}}:r\circ t_f=t_g\circ e} \mathsf{ap}_{t_g\circ -}(H_{\mathcal{L}})\cdot\mathsf{assoc}(t_g,e,s_f)^{-1}\cdot\mathsf{ap}_{-\circ s_f}(H_{\mathcal{R}})^{-1}\cdot\mathsf{assoc}(r,t_f,s_f)\cdot\mathsf{ap}_{r\circ -}(p_f)=\mathsf{assoc}(t_g,s_g,l)^{-1}\cdot\mathsf{ap}_{-\circ l}(p_g)\cdot S$$

$$\wr|$$

$$\sum_{I:\mathsf{Ob}(\mathcal{C})} \sum_{a_1:\mathsf{hom}_{\mathcal{C}}(A,I)} \sum_{a_2:\mathsf{hom}_{\mathcal{C}}(I,X)} \sum_{p_f:a_2\circ a_1=f} \sum_{b_1:\mathsf{hom}_{\mathcal{C}}(B,I)} \sum_{b_2:\mathsf{hom}_{\mathcal{C}}(I,Y)} \sum_{p_g:b_2\circ b_1=g} \sum_{H_{\mathcal{L}}:b_1\circ l=\mathsf{id}\circ a_1} \sum_{H_{\mathcal{R}}:r\circ a_2=b_2\circ\mathsf{id}} \sum_{a_1,b_1\in\mathcal{L}} \sum_{a_2,b_2\in\mathcal{R}} \mathsf{ap}_{b_2\circ -}(H_{\mathcal{L}})\cdot\mathsf{assoc}(b_2,\mathsf{id},a_1)^{-1}\cdot\mathsf{ap}_{-\circ a_1}(H_{\mathcal{R}})^{-1}\cdot\mathsf{assoc}(r,a_2,a_1)\cdot\mathsf{ap}_{r\circ -}(p_f)=\mathsf{assoc}(b_2,b_1,l)^{-1}\cdot\mathsf{ap}_{-\circ l}(p_g)\cdot S$$

$$\wr|$$

$$\sum_{I:\mathsf{Ob}(\mathcal{C})} \sum_{a_1:\mathsf{hom}_{\mathcal{C}}(A,I)} \sum_{a_2:\mathsf{hom}_{\mathcal{C}}(I,X)} \sum_{p_f:a_2\circ a_1=f} \sum_{b_1:\mathsf{hom}_{\mathcal{C}}(B,I)} \sum_{b_2:\mathsf{hom}_{\mathcal{C}}(I,Y)} \sum_{p_g:b_2\circ b_1=g} \sum_{H_{\mathcal{L}}:b_1\circ l=a_1} \sum_{H_{\mathcal{R}}:r\circ a_2=b_2} \sum_{a_1,b_1\in\mathcal{L}} \sum_{a_2,b_2\in\mathcal{R}} \mathsf{ap}_{b_2\circ -}(H_{\mathcal{L}})\cdot\mathsf{ap}_{b_2\circ -}(\mathsf{Lid}(a_1))^{-1}\cdot\mathsf{assoc}(b_2,\mathsf{id},a_1)^{-1}\cdot\mathsf{ap}_{-\circ a_1}(\mathsf{Rid}(b_2))\cdot\mathsf{ap}_{-\circ a_1}(H_{\mathcal{R}})^{-1}\cdot\mathsf{assoc}(r,a_2,a_1)\cdot\mathsf{ap}_{r\circ -}(p_f)=\mathsf{assoc}(b_2,a_2,l)^{-1}\cdot\mathsf{ap}_{-\circ l}(p_g)\cdot S$$

$$\wr|$$

$$\sum_{I:\mathsf{Ob}(\mathcal{C})} \sum_{a_2:\mathsf{hom}_{\mathcal{C}}(I,X)} \sum_{p_f:a_2\circ(b_1\circ l)=f} \sum_{b_1:\mathsf{hom}_{\mathcal{C}}(B,I)} \sum_{p_g:(r\circ a_2)\circ b_1=g} \sum_{b_1\in\mathcal{L}} \sum_{a_2\in\mathcal{R}} \mathsf{ap}_{b_2\circ -}(\mathsf{Lid}(b_1\circ l))^{-1}\cdot\mathsf{assoc}(b_2,\mathsf{id},a_1)^{-1}\cdot\mathsf{ap}_{-\circ a_1}(\mathsf{Rid}(r\circ a_2))\cdot\mathsf{assoc}(r,a_2,a_1)\cdot\mathsf{ap}_{r\circ -}(p_f)=\mathsf{assoc}(b_2,b_1,l)^{-1}\cdot\mathsf{ap}_{-\circ l}(p_g)\cdot S$$

$$\wr|$$

$$\sum_{I:\mathsf{Ob}(\mathcal{C})} \sum_{a_2:\mathsf{hom}_{\mathcal{C}}(I,X)} \sum_{b_1:\mathsf{hom}_{\mathcal{C}}(B,I)} \sum_{p_f:(a_2\circ b_1)\circ l=f} \sum_{p_g:r\circ(a_2\circ b_1)=g} \sum_{b_1\in\mathcal{L}} \sum_{a_2\in\mathcal{R}} \mathsf{ap}_{(r\circ a_2)\circ -}(\mathsf{Lid}(b_1\circ l))^{-1}\cdot\mathsf{assoc}(r\circ a_2,\mathsf{id},b_1\circ l)^{-1}\cdot\mathsf{ap}_{-\circ(b_1\circ l)}(\mathsf{Rid}(r\circ a_2))\cdot\mathsf{assoc}(r,a_2,b_1\circ l)\cdot\mathsf{ap}_{r\circ -}(\mathsf{assoc}(a_2,b_1,l))^{-1}\cdot\mathsf{ap}_{r\circ -}(p_f)=\mathsf{assoc}(r\circ a_2,b_1,l)^{-1}\cdot\mathsf{ap}_{-\circ l}(\mathsf{assoc}(r,a_2,b_1))\cdot\mathsf{ap}_{-\circ l}(p_g)\cdot S$$

$$\wr|$$

$$\sum_{I:\mathsf{Ob}(\mathcal{C})} \sum_{a_2:\mathsf{hom}_{\mathcal{C}}(I,X)} \sum_{b_1:\mathsf{hom}_{\mathcal{C}}(B,I)} \sum_{p_f:(a_2\circ b_1)\circ l=f} \sum_{p_g:r\circ(a_2\circ b_1)=g} \sum_{b_1\in\mathcal{L}} \sum_{a_2\in\mathcal{R}} \mathsf{ap}_{-\circ l}(\mathsf{assoc}(r,a_2,b_1))^{-1}\cdot\mathsf{assoc}(r\circ a_2,b_1,l)\cdot\underbrace{\mathsf{ap}_{(r\circ a_2)\circ -}(\mathsf{Lid}(b_1\circ l))^{-1}\cdot\mathsf{assoc}(r\circ a_2,\mathsf{id},b_1\circ l)^{-1}\cdot\mathsf{ap}_{-\circ(b_1\circ l)}(\mathsf{Rid}(r\circ a_2))}_{\mathsf{refl}_{(r\circ a_2)\circ(b_1\circ l)}}\cdot\mathsf{assoc}(r,a_2,b_1\circ l)\cdot\mathsf{ap}_{r\circ -}(\mathsf{assoc}(a_2,b_1,l))^{-1}\cdot\mathsf{ap}_{r\circ -}(p_f)=\mathsf{ap}_{-\circ l}(p_g)\cdot S$$

$$\wr|$$

$$\sum_{d:\mathsf{hom}_{\mathcal{C}}(B,X)} \underbrace{\sum_{I:\mathsf{Ob}(\mathcal{C})} \sum_{a_2:\mathsf{hom}_{\mathcal{C}}(I,X)} \sum_{b_1:\mathsf{hom}_{\mathcal{C}}(B,I)} \sum_{b_1\in\mathcal{L}} \sum_{a_2\in\mathcal{R}} \sum_{H_j:a_2\circ b_1=d}}_{\mathsf{fact}_{\mathcal{L},\mathcal{R}}(d)} \sum_{p_f:(a_2\circ b_1)\circ l=f} \sum_{p_g:r\circ(a_2\circ b_1)=g} \mathsf{assoc}(r,a_2\circ b_1,l)\cdot\mathsf{ap}_{r\circ -}(p_f)=\mathsf{ap}_{-\circ l}(p_g)\cdot S$$

$$\wr|$$

$$\mathsf{fill}(S)$$

It follows that $\mathsf{fill}(S)$ is contractible. ∎

Corollary 3.3.6. *We have that $\mathcal{L} = {}^{\perp}\mathcal{R}$ and $\mathcal{L}^{\perp} = \mathcal{R}$.*

Proof. We just prove that $\mathcal{L} = {}^{\perp}\mathcal{R}$ as the other case is dual. Let $f : \mathsf{hom}_{\mathcal{C}}(A, B)$. By Lemma 3.3.5, we know that $\mathcal{L}(f) \to {}^{\perp}\mathcal{R}(f)$. To prove the reverse implication, suppose that ${}^{\perp}\mathcal{R}(f)$. Factor f as $(\mathrm{im}(f), s_f, t_f, p_f)$ and consider the commuting square

$$\begin{array}{ccc} A & \xrightarrow{\;s_f\;} & \mathrm{im}(f) \\ {\scriptstyle f}\downarrow & \mathsf{Lid}(f) \cdot p_f^{-1} & \downarrow{\scriptstyle t_f} \\ B & \xrightarrow[\mathsf{id}]{} & B \end{array}$$

Since ${}^{\perp}\mathcal{R}(f)$, the type $\mathsf{fill}(\mathsf{Lid}(f) \cdot p_f^{-1})$ is contractible, with center, say, $\big(d, H_{s_f}, H_{\mathsf{id}}, K\big)$. The square

$$\begin{array}{ccc} A & \xrightarrow{\;s_f\;} & \mathrm{im}(f) \\ {\scriptstyle s_f}\downarrow & \mathsf{refl}_{t_f \circ s_f} & \downarrow{\scriptstyle t_f} \\ \mathrm{im}(f) & \xrightarrow[t_f]{} & B \end{array}$$

is contractible and has two diagonal fillers:

$$(\mathsf{id}, \mathsf{Lid}(s_f), \mathsf{Rid}(t_f), \ldots) \qquad \Big(d \circ t_f, \mathsf{assoc}(d, t_f, s_f) \cdot \mathsf{ap}_{d\circ -}(p_f) \cdot H_{s_f}, \mathsf{assoc}(t_f, d, t_f)^{-1} \cdot \mathsf{ap}_{-\circ t_f}(H_{\mathsf{id}}) \cdot \mathsf{Lid}(t_f), \omega\Big)$$

where ω denotes the chain of equalities

$$\begin{array}{c}
\mathsf{assoc}(t_f, d \circ t_f, s_f) \cdot \mathsf{ap}_{t_f \circ -}(\mathsf{assoc}(d, t_f, s_f) \cdot \mathsf{ap}_{d \circ -}(p_f) \cdot H_{s_f}) \\
\| \\
\mathsf{assoc}(t_f, d \circ t_f, s_f) \cdot \mathsf{ap}_{t_f \circ -}(\mathsf{assoc}(d, t_f, s_f)) \cdot \mathsf{ap}_{t_f \circ (d \circ -)}(p_f) \cdot \mathsf{ap}_{t_f \circ -}(H_{s_f}) \\
\|\,\textit{via } K \\
\mathsf{assoc}(t_f, d \circ t_f, s_f) \cdot \mathsf{ap}_{t_f \circ -}(\mathsf{assoc}(d, t_f, s_f)) \cdot \mathsf{ap}_{t_f \circ (d \circ -)}(p_f) \cdot \mathsf{assoc}(t_f, d, f)^{-1} \cdot \mathsf{ap}_{- \circ f}(H_{\mathsf{id}}) \cdot \mathsf{Lid}(f) \cdot p_f^{-1} \\
\|\,\textit{via the bicategory structure of } \mathcal{C} \\
\mathsf{ap}_{-\circ s_f}(\mathsf{assoc}(t_f, d, t_f))^{-1} \cdot \mathsf{assoc}(t_f \circ d, t_f, s_f) \cdot \mathsf{assoc}(t_f, d, t_f \circ s_f) \cdot \mathsf{ap}_{t_f \circ (d \circ -)}(p_f) \cdot \mathsf{assoc}(t_f, d, f)^{-1} \cdot \\
\mathsf{ap}_{-\circ f}(H_{\mathsf{id}}) \cdot \mathsf{Lid}(f) \cdot p_f^{-1} \cdot \mathsf{Lid}(t_f \circ s_f)^{-1} \cdot \mathsf{assoc}(\mathsf{id}, t_f, s_f)^{-1} \cdot \mathsf{ap}_{-\circ s_f}(\mathsf{Lid}(t_f)) \\
\|\,\textit{via homotopy naturality of } \mathsf{assoc}(t_f, d, -) \\
\mathsf{ap}_{-\circ s_f}(\mathsf{assoc}(t_f, d, t_f))^{-1} \cdot \mathsf{assoc}(t_f \circ d, t_f, s_f) \cdot \mathsf{ap}_{(t_f \circ d) \circ -}(p_f) \cdot \\
\mathsf{ap}_{-\circ f}(H_{\mathsf{id}}) \cdot \mathsf{ap}_{\mathsf{id} \circ -}(p_f)^{-1} \cdot \mathsf{assoc}(\mathsf{id}, t_f, s_f)^{-1} \cdot \mathsf{ap}_{-\circ s_f}(\mathsf{Lid}(t_f)) \\
\|\,\textit{via homotopy naturality of } p_f \\
\mathsf{ap}_{-\circ s_f}(\mathsf{assoc}(t_f, d, t_f))^{-1} \cdot \mathsf{assoc}(t_f \circ d, t_f, s_f) \cdot \mathsf{ap}_{-\circ (t_f \circ s_f)}(H_{\mathsf{id}}) \cdot \mathsf{assoc}(\mathsf{id}, t_f, s_f)^{-1} \cdot \mathsf{ap}_{-\circ s_f}(\mathsf{Lid}(t_f)) \\
\|\,\textit{via homotopy naturality of } \mathsf{assoc}(-, t_f, s_f) \\
\mathsf{ap}_{-\circ s_f}(\mathsf{assoc}(t_f, d, t_f))^{-1} \cdot \mathsf{ap}_{(-\circ t_f) \circ s_f}(H_{\mathsf{id}}) \cdot \mathsf{ap}_{-\circ s_f}(\mathsf{Lid}(t_f)) \\
\| \\
\mathsf{ap}_{-\circ s_f}(\mathsf{assoc}(t_f, d, t_f)^{-1} \cdot \mathsf{ap}_{-\circ t_f}(H_{\mathsf{id}}) \cdot \mathsf{Lid}(t_f))
\end{array}$$

It follows that d is an equivalence with inverse t_f. Thus, $d \in \mathcal{L}$. $\square$

Lemma 3.3.7. *Let $(\mathcal{L}, \mathcal{R})$ be an OFS on the category $\mathcal{U}$ of types. Consider a pushout square*

$$\begin{CD} C @>{g}>> B \\ @V{f}VV @VV{\mathsf{inr}}V \\ A @>>{\mathsf{inl}}> A \sqcup_C B \end{CD}$$

If f belongs to $\mathcal{L}$, then so does inr.

Proof. We must prove that for each lifting problem

$$\begin{CD} B @>{t}>> E \\ @V{\mathsf{inr}}VV S @VV{v}V \\ A \sqcup_C B @>>{b}> H \end{CD}$$

with $v \in \mathcal{R}$, the type $\mathsf{fill}(S)$ is contractible. Note that the type of fillers

$$\Phi \;:=\; \mathsf{fill}(\lambda x.\, \mathsf{ap}_b(\mathsf{glue}(x)) \cdot S(g(x)))$$

for the composite diagram

$$\begin{CD} C @>{g}>> B @>{t}>> E \\ @V{f}VV @. @VV{v}V \\ A @>>{\mathsf{inl}}> A \sqcup_C B @>>{b}> H \end{CD}$$

is contractible because $f \in \mathcal{L}$. By the induction principle for pushouts, we find that $\mathsf{fill}(S)$ is equivalent to the type of data

$$\begin{aligned} k \;&:\; A \to E \\ K_1 \;&:\; k \circ f \sim t \circ g \\ K_2 \;&:\; v \circ k \sim b \circ \mathsf{inl} \\ K_3 \;&:\; \prod_{x:C} \mathsf{ap}_v(K_1(x)) = K_2(f(x)) \cdot \mathsf{ap}_b(\mathsf{glue}(x)) \cdot S(g(x)) \end{aligned}$$

This type is exactly Φ, and thus $\mathsf{fill}(S)$ is contractible. $\square$

Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be 0-functors of wild categories. Suppose that $(\alpha, V_1, V_2) : L \dashv R$. Let $f : \mathsf{hom}_{\mathcal{C}}(A, B)$ and $g : \mathsf{hom}_{\mathcal{D}}(X, Y)$. Define

$$\begin{aligned} \varphi_\alpha \;&:\; \sum_{u:\mathsf{hom}_{\mathcal{D}}(LA,X)} \sum_{v:\mathsf{hom}_{\mathcal{D}}(LB,Y)} v \circ Lf = g \circ u \;\to\; \sum_{r:\mathsf{hom}_{\mathcal{C}}(A,RX)} \sum_{s:\mathsf{hom}_{\mathcal{C}}(B,RY)} s \circ f = Rg \circ r \\ \varphi_\alpha(u, v, G) \;&:=\; \big(\alpha(u), \alpha(v), V_2(f, v) \cdot \mathsf{ap}_\alpha(G) \cdot V_1(g, u)^{-1}\big)\,. \end{aligned}$$

Lemma 3.3.8. *The function φ_α is an equivalence, so that* $\mathsf{fill}(G) \simeq \mathsf{fill}(V_2(f,v) \cdot \mathsf{ap}_\alpha(G) \cdot V_1(g,u)^{-1})$ *for each $G : v \circ Lf = g \circ u$.*

Proof. Let $(r, s, H) : \sum_{r:\mathsf{hom}_\mathcal{C}(A,RX)} \sum_{s:\mathsf{hom}_\mathcal{C}(B,RY)} s \circ f = Rg \circ r$. We have that

$$\begin{array}{c} \mathsf{fib}_{\varphi_\alpha}(r,s,H) \\ \wr\! \mid \\ \sum_{u:\mathsf{hom}_\mathcal{D}(LA,X)} \sum_{e_1:\alpha(u)=r} \sum_{v:\mathsf{hom}_\mathcal{D}(LB,Y)} \sum_{e_2:\alpha(v)=s} \sum_{G:v\circ Lf=g\circ u} \mathsf{ap}_\alpha(G) = V_2(f,v)^{-1} \cdot \mathsf{ap}_{-\circ f}(e_2) \cdot H \cdot \mathsf{ap}_{Rg\circ -}(e_1)^{-1} \cdot V_1(g,u) \end{array}$$

This type is contractible because α is an equivalence. □

Corollary 3.3.9. *Suppose that both $\mathcal{C}$ and $\mathcal{D}$ are univalent bicategories endowed with OFS's $(\mathcal{L}_1, \mathcal{R}_1)$ and $(\mathcal{L}_2, \mathcal{R}_2)$, respectively. The 0-functor R preserves $\mathcal{R}$ if and only if L preserves $\mathcal{L}$.*

Proof. Suppose that R preserves $\mathcal{R}$. Let $f : \mathsf{hom}_\mathcal{C}(A,B)$ and $f \in \mathcal{L}_1$. Consider a commuting square

$$\begin{array}{ccc} LA & \xrightarrow{u} & X \\ {\scriptstyle Lf}\downarrow & S & \downarrow{\scriptstyle g} \\ LB & \xrightarrow[v]{} & Y \end{array}$$

where $g \in \mathcal{R}_2$. By Corollary 3.3.6, if $\mathsf{fill}(S)$ is contractible, then $Lf \in \mathcal{L}_2$. By Lemma 3.3.8, this type is equivalent to the type of fillers of the square

$$\begin{array}{ccc} A & \xrightarrow{\alpha(u)} & RX \\ {\scriptstyle f}\downarrow & V_2(f,v)\cdot\mathsf{ap}_\alpha(G)\cdot V_1(g,u)^{-1} & \downarrow{\scriptstyle Rg} \\ B & \xrightarrow[\alpha(v)]{} & RY \end{array}$$

By Corollary 3.3.6 again, this is contractible because $Rg \in \mathcal{R}_1$.

The converse is formally dual. □

3.4 Coslices of a universe

Let $\mathcal{U}$ be a universe and $A : \mathcal{U}$. Let $X, Y : A/\mathcal{U} :\equiv \sum_{X:\mathcal{U}} (A \to X)$. Consider the type

$$X \to_A Y \;:\equiv\; \sum_{h:\mathsf{pr}_1(X)\to\mathsf{pr}_1(Y)} h \circ \mathsf{pr}_2(X) \sim \mathsf{pr}_2(Y)$$

of maps from X to Y. In particular, note that

$$X \to_{\mathbf{1}} Y \quad = \quad (\mathsf{pr}_1(X), \mathsf{pr}_2(X)(*)) \to_* (\mathsf{pr}_1(Y), \mathsf{pr}_2(Y)(*))$$

the type of pointed maps from X to Y. For all $g : X \to_A Y$ and $h : Y \to_A Z$. define their composite

$$h \circ g :\equiv \Big(\mathsf{pr}_1(h) \circ \mathsf{pr}_1(g), \lambda a.\, \mathsf{ap}_{\mathsf{pr}_1(h)}(\mathsf{pr}_2(g)(a)) \cdot \mathsf{pr}_2(h)(a)\Big) \;:\; X \to_A Z$$

This gives us a wild category $A/\mathcal{U}$, called the *coslice of* $\mathcal{U}$ *under* A.

Example 3.4.1. The coslice $\mathbf{1}/\mathcal{U}$ is called the category of *pointed types*, sometimes denoted by $\mathcal{U}^*$.

Proposition 3.4.2. *For all* $X, Y : A/\mathcal{U}$*, we have an equivalence*

$$(X = Y) \simeq \sum_{k:\mathsf{pr}_1(X) \xrightarrow{\simeq} \mathsf{pr}_1(Y)} k \circ \mathsf{pr}_2(X) \sim \mathsf{pr}_2(Y)$$

Definition 3.4.3. Let $f, g : X \to_A Y$. An A*-homotopy* $f \sim_A g$ between f and g is a homotopy $H : \mathsf{pr}_1(f) \sim \mathsf{pr}_1(g)$ along with a path $H(\mathsf{pr}_2(X)(a))^{-1} \cdot \mathsf{pr}_2(f)(a) = \mathsf{pr}_2(g)(a)$ for all $a : A$.

Lemma 3.4.4. *The canonical function* $\mathsf{happly}_A : (f = g) \to (f \sim_A g)$ *is an equivalence For all* f, g.

Proof. By Theorem A.0.3. □

Notation. Define $\langle H, p \rangle :\equiv \mathsf{happly}_A^{-1}(H, p)$.

Inherited reflective subcategories

Let $(P, \bigcirc, \eta)$ be a reflective subcategory of $\mathcal{U}$ (see Definition 3.2.1). Then the data

$$\begin{aligned} P_A(X) &:\equiv P(\mathsf{pr}_1(X)) \\ \bigcirc_A(X) &:\equiv \big(\bigcirc(\mathsf{pr}_1(X)), \eta_{\mathsf{pr}_1(X)} \circ \mathsf{pr}_2(X)\big) \\ \eta_A(X) &:\equiv \Big(\eta_{\mathsf{pr}_1(X)}, \lambda a.\mathsf{refl}_{\eta_{\mathsf{pr}_1(X)}(\mathsf{pr}_2(X)(a))}\Big) \end{aligned}$$

is a reflective subcategory of $A/\mathcal{U}$, denoted by $(A/\mathcal{U})_P$. Indeed, for all Y with $P(\mathsf{pr}_1(Y))$, the maps

$$\begin{aligned} \alpha_{P,A} &: (\bigcirc_A X \to_A Y) \to (X \to_A Y) \\ \alpha_{P,A}(f, f_p) &:\equiv \big(f \circ \eta_{\mathsf{pr}_1(X)}, f_p\big) \\ \mathsf{rec}_{P,A} &: (X \to_A Y) \to (\bigcirc_A X \to_A Y) \\ \mathsf{rec}_{P,A}(g, g_p) &:\equiv (\mathsf{rec}_{\bigcirc}(g), \lambda a.\beta_\eta(g, \mathsf{pr}_2(X)(a)) \cdot g_p(a)) \end{aligned}$$

are mutual inverses. (Recall that β_η has type $\prod_{X,Y:\mathcal{U}} \prod_{\_:P(Y)} \prod_{g:X \to Y} \mathsf{rec}_{\bigcirc}(g) \circ \eta_{\mathsf{pr}_1(X)} \sim g$.)

Note 3.4.5. The function $\bigcirc_A$ induces an action on maps in $A/\mathcal{U}$. For every $m : X_1 \to_A X_2$, define $\bigcirc_A(m) :\equiv \mathsf{rec}_{P,A}(\eta_A(X_2) \circ m) : \bigcirc_A(X_1) \to \bigcirc_A(X_2)$. We claim that $\alpha_{P,A}$ is natural in X. For all

$(h, h_p) : X_2 \to_A X_1$ and $(f, f_p) : \bigcirc_A X_1 \to_A Y$,

$$\begin{array}{c} \alpha_{P,A}(f, f_p) \circ (h, h_p) \\ \| \\ \textit{definitional} \\ \| \\ \left(f \circ \eta_{\mathsf{pr}_1(X_1)} \circ h, \lambda a.\, \mathsf{ap}_{f \circ \eta_{\mathsf{pr}_1(X_1)}}(h_p(a)) \cdot f_p(a)\right) \\ \| \\ {\scriptstyle \left\langle \lambda x.\, \mathsf{ap}_f(\beta_\eta(\eta_{\mathsf{pr}_1(X_1)} \circ h, x))^{-1}, \lambda a.\, \mathsf{PI}(\beta_\eta(\eta_{\mathsf{pr}_1(X_1)} \circ h, \mathsf{pr}_2(X_2)(a)), h_p(a))\right\rangle} \\ \| \\ \left(f \circ \mathsf{rec}_{\bigcirc}(\eta_{\mathsf{pr}_1(X_1)} \circ h) \circ \eta_{\mathsf{pr}_1(X_2)}, \lambda a.\, \mathsf{ap}_f(\beta(\eta_{\mathsf{pr}_1(X_1)} \circ h, \mathsf{pr}_2(X_2)(a)) \cdot \mathsf{ap}_{\eta_{\mathsf{pr}_1(X_1)}}(h_p(a)) \cdot \mathsf{refl}_{\eta_{\mathsf{pr}_1(X_1)}(\mathsf{pr}_2(X_1)(a))}) \cdot f_p(a)\right) \\ \| \\ \textit{definitional} \\ \| \\ \alpha_{P,A}((f, f_p) \circ \bigcirc_A(h, h_p)) \end{array}$$

Proposition 3.4.6. *The left adjoint $\|-\|_n : A/\mathcal{U} \to (A/\mathcal{U})_{\leq n}$ is 2-coherent (Definition B.0.1).*

Corollary 3.4.7. *The functor $\|-\|_n : A/\mathcal{U} \to (A/\mathcal{U})_{\leq n}$ preserves colimits.*

Proof. By Theorem B.0.2. ☐

3.5 Diagrams in coslices

Let Γ be a graph. An *A-diagram over* Γ is a family $F : \Gamma_0 \to A/\mathcal{U}$ of objects in $A/\mathcal{U}$ along with a map $F_{i,j,g} : F_i \to_A F_j$ for all $i, j : \Gamma_0$ and $g : \Gamma_1(i, j)$.

Example 3.5.1. For each $D : \mathsf{Ob}(A/\mathcal{U})$, the *constant diagram* $\mathsf{const}_\Gamma(D)$ *at* D is defined by $(\mathsf{const}_\Gamma(D))_0\,(i) :\equiv D$ and $(\mathsf{const}_\Gamma(D))_1\,(i, j, g) :\equiv \mathsf{id}_D$. We often write just D for $\mathsf{const}_\Gamma(D)$.

Let F be an A-diagram over Γ and $C : A/\mathcal{U}$. A *cocone under F on C* is a term $r : \prod_i F_i \to_A C$ with

- for each $i, j : \Gamma_0$ and $g : \Gamma_1(i, j)$, a homotopy $h_{i,j,g} : \mathsf{pr}_1(r_j) \circ \mathsf{pr}_1(F_{i,j,g}) \sim \mathsf{pr}_1(r_i)$
- for each $a : A$, a path $h_{i,j,g}(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a) \;=\; \mathsf{ap}_{\mathsf{pr}_1(r_i)}(a)$

Let $\mathsf{Cocone}_F(C)$ denote the type of cocones under F on C.

Lemma 3.5.2. *For all $(\alpha, p), (\beta, q) : \mathsf{Cocone}_F(C)$, $(\alpha, p) = (\beta, q)$ is equivalent to the type of data*

$$W \;:\; \prod_{i:\Gamma_0} \mathsf{pr}_1(\alpha_i) \sim \mathsf{pr}_1(\beta_i)$$

$$u \;:\; \prod_{i:\Gamma_0} \prod_{a:A} W_i(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{pr}_2(\alpha_i)(a) = \mathsf{pr}_2(\beta_i)(a)$$

for all $i, j : \Gamma_0,\ g : \Gamma_1(i, j)\ \ \ldots$

$$S_1(i, j, g) \;:\; \prod_{x:\mathsf{pr}_1(F_i)} W_j(\mathsf{pr}_1(F_{i,j,g})(x))^{-1} \cdot \mathsf{pr}_1(p_{i,j,g})(x) \cdot W_i(x) = \mathsf{pr}_1(q_{i,j,g})(x)$$

$$S_2(i, j, g) \;:\; \prod_{a:A} \mathsf{ap}_{-^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\beta_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(\beta_j)(a)}(S_1(i, j, g, \mathsf{pr}_2(F_i)(a)))^{-1} \cdot \Xi(W, u, p_{i,j,g}, a) = \mathsf{pr}_2(q_{i,j,g})(a)$$

Here, $\Xi(W, u, p_{i,j,g}, a)$ denotes the chain of equalities shown on the next page.

$$\big(W_j(\mathsf{pr}_1(F_{i,j,g})(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a)) \cdot W_i(\mathsf{pr}_2(F_i)(a))\big)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\beta_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(\beta_j)(a)$$

$$\Big\Vert \; \mathsf{PI}(\mathsf{pr}_2(F_{i,j,g})(a), W_j(\mathsf{pr}_2(F_j)(a)))$$

$$\Big(\Big(\mathsf{ap}_{\mathsf{pr}_1(\beta_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot W_j(\mathsf{pr}_2(F_j)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\alpha_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}\Big) \cdot \mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a)) \cdot W_i(\mathsf{pr}_2(F_i)(a))\Big)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\beta_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(\beta_j)(a)$$

$$\Big\Vert \; \textit{via } u_j(a)$$

$$\Big(\Big(\mathsf{ap}_{\mathsf{pr}_1(\beta_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \big(\mathsf{pr}_2(\beta_j)(a) \cdot \mathsf{pr}_2(\alpha_j)(a)^{-1}\big) \cdot \mathsf{ap}_{\mathsf{pr}_1(\alpha_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}\Big) \cdot \mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a)) \cdot W_i(\mathsf{pr}_2(F_i)(a))\Big)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\beta_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(\beta_j)(a)$$

$$\Big\Vert \; \mathsf{PI}(\mathsf{pr}_2(F_{i,j,g})(a), \mathsf{pr}_2(\beta_j)(a), \mathsf{pr}_2(\alpha_j)(a), \mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a)), W_i(\mathsf{pr}_2(F_i)(a)))$$

$$W_i(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(\alpha_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(\alpha_j)(a)$$

$$\Big\Vert \; \mathsf{ap}_{W_i(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \_}(\mathsf{pr}_2(p_{i,j,g})(a))$$

$$W_i(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{pr}_2(\alpha_i)(a)$$

$$\Big\Vert \; u_i(a)$$

$$\mathsf{pr}_2(\beta_i)(a)$$

Proof. By Theorem A.0.3. □

Note 3.5.3.

(a) For all A-diagrams F and G over a graph Γ, the type of *natural transformations* from F to G is

$$F \Rightarrow_A G \quad :\equiv \sum_{\alpha:\prod_{i:\Gamma_0} \mathsf{pr}_1(F_i)\to_A \mathsf{pr}_1(G_i)} \prod_{i,j:\Gamma_0} \prod_{g:\Gamma_1(i,j)} (G_{i,j,g} \circ \alpha_i \sim_A \alpha_j \circ F_{i,j,g})$$

For each $C : A/\mathcal{U}$, we have an evident equivalence $\mathsf{Cocone}_F(C) \;\simeq\; (F \Rightarrow_A \mathsf{const}_\Gamma(C))$.

(b) For every graph Γ and every $\mathcal{U}$-valued diagram F over Γ, recall the (standard) limit of F [2, Definition 4.2.7]:

$$\mathsf{lim}_\Gamma(F) \;:\equiv \sum_{\alpha:\prod_{i:\Gamma_0} F_i} \prod_{i,j:\Gamma_0} \prod_{g:\Gamma_1(i,j)} F_{i,j,g}(\alpha_i) =_{F_j} \alpha_j$$

This is functorial in F. The action on maps sends $(k, K) : F \Rightarrow G$ to the function $\mathsf{lim}_\Gamma(k, K) : \mathsf{lim}_\Gamma(F) \to \mathsf{lim}_\Gamma(G)$ defined by $(\alpha, D) \mapsto \Big(\lambda i.k_i(\alpha_i), \lambda i\lambda j\lambda g.D_{i,j,g}(\alpha_i) \cdot \mathsf{ap}_{k_j}(K_{i,j,g})\Big)$.

Let F be an A-diagram over a graph Γ and let $C : A/\mathcal{U}$. We have an evident equivalence $\mathsf{Cocone}_F(C) \;\simeq\; \mathsf{lim}_{i:\Gamma^{\mathrm{op}}}(F_i \to_A C)$.

4 Trees

A *(directed) graph* Γ is a pair consisting of a type $\Gamma_0 : \mathcal{U}$ and a family $\Gamma_1 : \Gamma_0 \to \Gamma_0 \to \mathcal{U}$ of types.

Definition 4.0.1. Let Γ be a graph.

1. The *geometric realization* $|\Gamma| :\equiv {}^{\Gamma_0}/_{\Gamma_1}$ of Γ is the HIT generated by
   - $|-| : \Gamma_0 \to {}^{\Gamma_0}/_{\Gamma_1}$
   - $\mathsf{quot\text{-}rel} : \prod_{x,y:\Gamma_0} \Gamma_1(x, y) \to |x| = |y|$.
2. We say that Γ is a *tree* if $|\Gamma|$ is contractible.

In Section 5.4, we will see that A-colimits interact nicely with *trees*.

Example 4.0.2.

- Both $\mathbb{N}$ and $\mathbb{Z}$ are trees when viewed as graphs.
- The span $l \leftarrow m \to r$ is a tree where l, m, r denote the elements of $\mathsf{Fin}(3)$.

Let Γ be a graph. Define the indexed inductive type of *walks from i to j in* Γ

$$\begin{aligned}
\mathcal{W}_\Gamma &: \Gamma_0 \to \Gamma_0 \to \mathcal{U} \\
\mathsf{nil} &: \prod_{i:\Gamma_0} \mathcal{W}_\Gamma(i,i) \\
\mathsf{cons} &: \prod_{i,j,k:\Gamma_0} \Gamma_1(i,j) \to \mathcal{W}_\Gamma(j,k) \to \mathcal{W}_\Gamma(i,k)
\end{aligned}$$

Definition 4.0.3. Let $j_0 : \Gamma_0$. We say that Γ is a *combinatorial tree* if

- for every $i : \Gamma_0$, we have a term $\nu(i, j_0) : \mathcal{W}_\Gamma(i, j_0)$
- for every $i, j : \Gamma_0$ and $g : \Gamma_1(i,j)$, we have a term $\sigma_g : \nu(i, j_0) = \mathsf{cons}(g, \nu(j, j_0))$

Lemma 4.0.4. *For every $i, j : \Gamma_0$ and $z : \mathcal{W}_\Gamma(i,j)$, we have a term $\tau(z) : |i| = |j|$.*

Proof. We proceed by induction on $\mathcal{W}_\Gamma$.

- Define $\tau(\mathsf{nil}_i) :\equiv \mathsf{refl}_{|i|}$.
- Let $i, j, k : \Gamma_0$, $g : \Gamma_1(i,j)$, and $z : \mathcal{W}_\Gamma(j,k)$. Suppose that $\tau(z)$ has been defined. Define $\tau(\mathsf{cons}(g, z)) :\equiv \mathsf{quot\text{-}rel}(g) \cdot \tau(z)$.

□

Lemma 4.0.5. *Every combinatorial tree is a tree.*

Proof. Let Γ be a combinatorial tree. It suffices to prove that for every $x : {}^{\Gamma_0}/_{\Gamma_1}$, $x = |j_0|$. We proceed by induction on quotients. For each $i : \Gamma_0$, we have the term $\tau(\nu(i, j_0)) : |i| = |j_0|$ by Lemma 4.0.4. Since Γ is a combinatorial tree, we also see that for all $i, j : \Gamma_0$ and $g : \Gamma_1(i,j)$,

$$\begin{aligned}
&\mathsf{quot\text{-}rel}(g)_*(\tau(\nu(i, j_0))) \\
=\ &\mathsf{quot\text{-}rel}(g)^{-1} \cdot \tau(\nu(i, j_0)) \\
=\ &\mathsf{quot\text{-}rel}(g)^{-1} \cdot \tau(\mathsf{cons}(g, \nu(j, j_0))) \\
\equiv\ &\mathsf{quot\text{-}rel}(g)^{-1} \cdot \mathsf{quot\text{-}rel}(g) \cdot \tau(j, \nu_{j_0,i}) \\
=\ &\tau(j, \nu_{j_0,i})
\end{aligned}$$

This completes the induction proof. □

Corollary 4.0.6. *Every directed tree, in the sense of [12, Directed trees], is a tree.*

Proof. Just notice that every directed tree is a combinatorial tree. □

Example 4.0.7. Trees are abundant in HoTT. Indeed, consider a coalgebra for a polynomial endofunctor $\mathcal{P}_{A,B}$ for a signature (A, B):

$$\mathcal{X} :\equiv \left(X,\ \alpha : X \to \sum_{a:A} (B(a) \to X) \right)$$

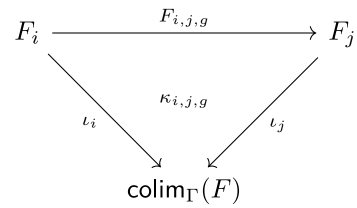
All elements of X can be made into a directed tree [12, The underlying trees of elements of coalgebras of polynomial endofunctors]. Hence every element of the W-type for (A, B) is a tree as $\mathsf{W}(A, B)$ has a canonical coalgebra structure [12, W-types as coalgebras for a polynomial endofunctor]. Also, every element of the coinductive type $\mathsf{M}(A, B)$, the terminal coalgebra for $\mathcal{P}_{A,B}$, is a tree.

5 Colimits

5.1 Ordinary colimits

The colimit $\mathsf{colim}_\Gamma(F)$ of a diagram F in $\mathcal{U}$ over Γ is the HIT generated by

$$\begin{aligned} \iota &: (i : \Gamma_0) \to F_i \to \mathsf{colim}_\Gamma(F) \\ \kappa &: (i, j : \Gamma_0)\,(g : \Gamma_1(i, j)) \to \iota_j \circ F_{i,j,g} \sim \iota_i \end{aligned}$$



We have the following induction principle.

$$\begin{aligned} E &: \mathsf{colim}_\Gamma(F) \to \mathcal{U} \\ e &: \prod_{i:\Gamma_0} \prod_{x:F_i} E(\iota_i(x)) \\ q &: \prod_{i,j:\Gamma_0} \prod_{g:\Gamma_1(i,j)} \prod_{x:F_i} \mathsf{transp}^E(\kappa_{i,j,g}(x), e_j(F_{i,j,g}(x))) = e_i(x) \\ &\Downarrow \\ \mathsf{ind}(E, e, q) &: \prod_{z:\mathsf{colim}_\Gamma(F)} E(z), \quad \mathsf{ind}(E, e, q)(\iota_i(x)) \equiv e_i(x) \\ \rho_e(q) &: \prod_{i,j:\Gamma_0} \prod_{g:\Gamma_1(i,j)} \prod_{x:F_i} \mathsf{apd}_{\mathsf{ind}(E,e,q)}(\kappa_{i,j,g}(x)) = q_{i,j,g}(x) \end{aligned}$$

along with a recurison principle $\mathsf{rec}(E, e, q)$ in the non-dependent case.

Example 5.1.1.

1. If $\Gamma_0 \equiv \mathbb{N}$ and $\Gamma_1(i, j) \equiv\ i + 1 = j$, then Γ is precisely the ordinal ω. (We may abuse notation by referring to ω as just $\mathbb{N}$.) For every type family $F : \mathbb{N} \to \mathcal{U}$, we have an equivalence

$$\begin{aligned} \epsilon &: \left(\prod_{n,m:\mathbb{N}} (n + 1 = m) \to F_n \to F_m \right) \xrightarrow{\simeq} \left(\prod_{n:\mathbb{N}} F_n \to F_{n+1} \right) \\ \epsilon(F, n) &:\equiv F_{n,n+1}(\mathsf{refl}_{n+1}) \end{aligned}$$

   along with an equivalence $\mathsf{colim}(F) \simeq \overbrace{\mathsf{colim}_{\mathsf{seq}}(\epsilon(F))}^{\text{sequential colimit}}$ for every diagram F over ω. Specifically, construct a quasi-inverse of ϵ by sending each $f : \prod_{n:\mathbb{N}} F_n \to F_{n+1}$ to

$$\lambda n \lambda m \lambda g.\, \mathsf{transp}^{k \mapsto F_n \to F_k}(g, f_n) \;:\; \prod_{n,m:\mathbb{N}} (n+1=m) \to F_n \to F_m$$

2. Suppose $\Gamma_0 \equiv \{l, r, m\}$ (i.e., $\mathsf{Fin}(3)$) and $\Gamma_1(m,l) \equiv \mathbf{1}$, $\Gamma_1(m,r) \equiv \mathbf{1}$, and $\Gamma_1(i,j) \equiv \mathbf{0}$ otherwise. Then $\mathsf{colim}(F)$ is equivalent to the pushout $F(l) \sqcup_{F(m)} F(r)$, i.e., the HIT generated by
   - $\mathsf{left} : F(l) \to F(l) \sqcup_{F(m)} F(r)$
   - $\mathsf{right} : F(r) \to F(l) \sqcup_{F(m)} F(r)$
   - $\mathsf{glue} : \prod_{x:F(m)} (\mathsf{left}(F_{m,l}(*)(x)) = \mathsf{right}(F_{m,r}(*)(x)))$.
3. If Γ_0 is a type and $\Gamma_1(i,j) \equiv \mathbf{0}$ for all $i, j : \Gamma_0$, then Γ is called the *discrete graph on* Γ_0. In this case, $\mathsf{colim}_\Gamma(F)$ is equivalent to the coproduct $\sum_{i:\Gamma_0} F_i$.

Lemma 5.1.2. *For every graph* Γ*,* $\mathsf{colim}_\Gamma\, \mathbf{1} \simeq {}^{\Gamma_0}/_{\Gamma_1}$.

Corollary 5.1.3. *Let* Γ *be a tree and* $A : \mathcal{U}$*. The function* $[\mathsf{id}_A]_{i:\Gamma_0} : \mathsf{colim}_\Gamma A \to A$ *is an equivalence.*

Proof. Suppose that Γ is a tree. As $A \times -$ has right adjoint $A \to -$, the following diagram commutes:

$$\mathsf{colim}_\Gamma A \xrightarrow{\simeq} \mathsf{colim}_\Gamma(A \times \mathbf{1}) \xrightarrow{\simeq} A \times \mathsf{colim}_\Gamma\, \mathbf{1} \xrightarrow{\simeq} A, \qquad \mathsf{colim}_\Gamma A \xrightarrow{[\mathsf{id}_A]} A$$

□

Lemma 5.1.4. *Let* Γ *be a graph. Suppose that* F *is a diagram over* Γ*. Let* Z *be a type and* $h_1, h_2 : \mathsf{colim}_\Gamma(F) \to Z$*. If we have a term* $p_i(x) : h_1(\iota_i(x)) = h_2(\iota_i(x))$ *for all* $i : \Gamma_0$ *and* $x : F_i$ *along with a commuting square*

$$\begin{array}{ccc} h_1(\iota_j(F_{i,j,g}(x))) & \xlongequal{\mathsf{ap}_{h_1}(\kappa_{i,j,g}(x))} & h_1(\iota_i(x)) \\ {\scriptstyle p_j(F_{i,j,g}(x))}\Big\Downarrow & & \Big\Downarrow{\scriptstyle p_i(x)} \\ h_2(\iota_j(F_{i,j,g}(x))) & \xRightarrow[\mathsf{ap}_{h_2}(\kappa_{i,j,g}(x))]{} & h_2(\iota_i(x)) \end{array}$$

for all $i, j : \Gamma_0$*,* $g : \Gamma_1(i,j)$*, and* $x : F_i$*, then* $h_1 \sim h_2$*.*

Proof. By induction on $\mathsf{colim}_\Gamma(F)$. □

Lemma 5.1.5. *Consider a pushout square*

$$\begin{array}{ccc} C & \xrightarrow{g} & B \\ {\scriptstyle f}\big\downarrow & \mathsf{glue} & \big\downarrow{\scriptstyle \mathsf{inr}} \\ A & \xrightarrow[\mathsf{inl}]{} & A \sqcup_C B \end{array}$$

Let Z be a type and $h_1, h_2 : A \sqcup_C B \to Z$. If we have terms

$$p_1 \;:\; \prod_{a:A} h_1(\mathsf{inl}(a)) = h_2(\mathsf{inl}(a)) \qquad\qquad p_2 \;:\; \prod_{b:B} h_1(\mathsf{inr}(b)) = h_2(\mathsf{inr}(b))$$

along with a commuting square

$$\begin{array}{ccc}
h_1(\mathsf{inl}(f(c))) & \xRightarrow{\mathsf{ap}_{h_1}(\mathsf{glue}(c))} & h_1(\mathsf{inr}(g(c))) \\
{\scriptstyle p_1(f(c))}\Big\Downarrow & & \Big\Downarrow{\scriptstyle p_2(g(c))} \\
h_2(\mathsf{inl}(f(c))) & \xRightarrow[\mathsf{ap}_{h_2}(\mathsf{glue}(c))]{} & h_2(\mathsf{inr}(g(c)))
\end{array}$$

of paths in Z for every $c : C$, then $h_1 \sim h_2$.

Proof. By pushout induction. □

Note that colim_Γ is a functor from the wild category of diagrams over Γ to $\mathcal{U}$. In particular, for each $(\alpha, p) : F \Rightarrow G$, the function $\mathsf{colim}(\alpha, p) : \mathsf{colim}_\Gamma(F) \to \mathsf{colim}_\Gamma(G)$ is the canonical map induced by the following cocone under F:

$$\begin{array}{ccccc}
F_i & \xrightarrow{\quad F_{i,j,g} \quad} & F_j & & \\
{\scriptstyle \alpha_i}\downarrow & & \downarrow{\scriptstyle \alpha_j} & & \\
G_i & \overset{G_{i,j,g}}{\cdots\cdots\rightarrow} & G_j & \qquad & (\lambda x.\, \mathsf{ap}_{\iota_j}(p_{i,j,g}(x))^{-1} \cdot \kappa^G_{i,j,g}(\alpha_i(x))) \\
{\scriptstyle \iota_i}\searrow & & \swarrow{\scriptstyle \iota_j} & & \\
& \mathsf{colim}_\Gamma(G) & & &
\end{array}$$

Moreover, the pushout HIT is a functor on spans. For each map (ψ, S) of spans

$$\begin{array}{ccccc}
A_1 & \xleftarrow{f_1} & C_1 & \xrightarrow{g_1} & B_1 \\
{\scriptstyle \psi_1}\downarrow & S_1 & \downarrow{\scriptstyle \psi_2} & S_2 & \downarrow{\scriptstyle \psi_3} \\
A_2 & \xleftarrow[f_2]{} & C_2 & \xrightarrow[g_2]{} & B_2
\end{array}$$

the function $\mathsf{po}(\psi, S) : A_1 \sqcup_{C_1} B_1 \to A_2 \sqcup_{C_2} B_2$ is the canonical map induced by

$$\begin{array}{ccccc}
C_1 & \xrightarrow{\quad g_1 \quad} & B_1 & & \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle \mathsf{inr} \circ \psi_3} & \qquad & (\lambda x.\, \mathsf{ap}_{\mathsf{inl}}(S_1(x))^{-1} \cdot \mathsf{glue}_2(\psi_2(x)) \cdot \mathsf{ap}_{\mathsf{inr}}(S_2(x))) \\
A_1 & \xrightarrow[\mathsf{inl} \circ \psi_1]{} & A_2 \sqcup_{C_2} B_2 & &
\end{array}$$

5.2 Coslice colimits

Let $A : \mathcal{U}$ and F be an A-diagram over a graph Γ. An A-cocone (C, r, p) under F is *colimiting* if

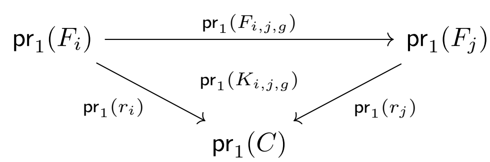
$$\begin{aligned}
&\mathsf{postcomp}(C, r, p, T) \;:\; (C \to_A T) \to \mathsf{Cocone}_F(T) \\
&\mathsf{postcomp}(C, r, p, T)(f, f_p) \;:\equiv \\
&\quad \Big(\lambda i.\, \big(f \circ \mathsf{pr}_1(r_i), \lambda a.\, \mathsf{ap}_f(\mathsf{pr}_2(r_i)(a)) \cdot f_p(a)\big)\,, \lambda i \lambda j \lambda g.\, \Big(\lambda x.\, \mathsf{ap}_f(\mathsf{pr}_1(p_{i,j,g})(x)), \lambda a. \Theta_{\mathsf{pr}_1(p_{i,j,g})}(f^*, a) \cdot \mathsf{ap}_{- \cdot f_p(a)}(\mathsf{ap}_{\mathsf{ap}_f}(\mathsf{pr}_2(p_{i,j,g})(a)))\Big)\Big)
\end{aligned}$$

is an equivalence for every $T : A/\mathcal{U}$, where $\Theta_{\mathsf{pr}_1(p_{i,j,g})}(f^*, a)$ has type

$$\begin{gathered}
\mathsf{ap}_f(\mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_{f \circ \mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_f(\mathsf{pr}_2(r_j)(a)) \cdot f_p(a) \\
\| \\
\mathsf{ap}_f(\mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)) \cdot f_p(a)
\end{gathered}$$

and is defined by double path induction on $\mathsf{pr}_1(p_{i,j,g})(\mathsf{pr}_2(F_i)(a))$ and $\mathsf{pr}_2(F_{i,j,g})(a)$.

Note 5.2.1 (Forgetful functor). The functor $\mathsf{pr}_1 : A/\mathcal{U} \to \mathcal{U}$ induces a functor $\mathcal{F} : \mathsf{Diag}_A(\Gamma) \to \mathsf{Diag}(\Gamma)$ from the category of diagrams in $A/\mathcal{U}$ to that of diagrams in $\mathcal{U}$. It also induces a functor $\mathcal{F} : \mathsf{Cocone}(F) \to \mathsf{Cocone}(\mathsf{pr}_1 \circ F)$ between categories of cocones for each diagram $F : \Gamma \to A/\mathcal{U}$. In this case, $\mathcal{F}$ maps a cocone (C, r, K) under F to the following cocone under $\mathcal{F}(F)$:



Let $\mathcal{A} :\equiv (C, m, M)$ and $\mathcal{B} :\equiv (B, k, K)$ be A-cocones under F. A morphism $\mathcal{A} \to \mathcal{B}$ consists of terms

$$\begin{aligned}
(f, p) \;&:\; C \to_A B \\
d \;&:\; \prod_{i:\Gamma_0} f \circ \mathsf{pr}_1(m_i) \sim \mathsf{pr}_1(k_i) \\
e \;&:\; \prod_{i:\Gamma_0} \prod_{a:A} d_i(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_f(\mathsf{pr}_2(m_i)(a)) \cdot p(a) = \mathsf{pr}_2(k_i)(a) \\
U \;&:\; \prod_{i,j,g} \prod_{x:\mathsf{pr}_1(F_i)} d_j(\mathsf{pr}_1(F_{i,j,g})(x))^{-1} \cdot \mathsf{ap}_f(\mathsf{pr}_1(M_{i,j,g})(x)) \cdot d_i(x) = \mathsf{pr}_1(K_{i,j,g})(x) \\
V \;&:\; \prod_{i,j,g} \prod_{a:A} \Lambda_{i,j,g}(a) = \mathsf{pr}_2(K_{i,j,g})(a)
\end{aligned}$$

where $\Lambda_{i,j,g}(a)$ denotes the chain of paths shown on the next page.

$$\mathsf{pr}_1(K_{i,j,g})(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(k_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(k_j)(a)$$

$$\Big\| \; \textit{via } U_{i,j,g}(\mathsf{pr}_2(F_i)(a))$$

$$\big(d_j(\mathsf{pr}_1(F_{i,j,g})(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_f(\mathsf{pr}_1(M_{i,j,g})(\mathsf{pr}_2(F_i)(a))) \cdot d_i(\mathsf{pr}_2(F_i)(a))\big)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(k_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(k_j)(a)$$

$$\Big\| \; \textit{homotopy naturality of } d_j \textit{ at } \mathsf{pr}_2(F_{i,j,g})(a)$$

$$\Big(\Big(\mathsf{ap}_{\mathsf{pr}_1(k_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot d_j(\mathsf{pr}_2(F_j)(a))^{-1} \cdot \mathsf{ap}_{f\circ\mathsf{pr}_1(m_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}\Big) \cdot \mathsf{ap}_f(\mathsf{pr}_1(M_{i,j,g})(\mathsf{pr}_2(F_i)(a))) \cdot d_i(\mathsf{pr}_2(F_i)(a))\Big)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(k_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(k_j)(a)$$

$$\Big\| \; \textit{via } e_j(a)$$

$$\Big(\Big(\mathsf{ap}_{\mathsf{pr}_1(k_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \Big(\mathsf{pr}_2(k_j)(a) \cdot \big(\mathsf{ap}_f(\mathsf{pr}_2(m_j)(a)) \cdot p(a)\big)^{-1}\Big) \cdot \mathsf{ap}_{f\circ\mathsf{pr}_1(m_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}\Big) \cdot \mathsf{ap}_f(\mathsf{pr}_1(M_{i,j,g})(\mathsf{pr}_2(F_i)(a))) \cdot d_i(\mathsf{pr}_2(F_i)(a))\Big)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(k_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(k_j)(a)$$

$$\Big\| \; \mathsf{PI}(\mathsf{pr}_2(F_{i,j,g})(a),\mathsf{pr}_2(k_j)(a),d_i(\mathsf{pr}_2(F_i)(a)),\mathsf{pr}_1(M_{i,j,g})(\mathsf{pr}_2(F_i)(a)),\mathsf{pr}_2(m_j)(a),p(a))$$

$$d_i(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_f(\mathsf{pr}_1(M_{i,j,g})(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(m_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(m_j)(a)) \cdot p(a)$$

$$\Big\| \; \textit{via } \mathsf{pr}_2(M_{i,j,g})(\mathsf{pr}_2(F_i)(a))$$

$$d_i(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_f(\mathsf{pr}_2(m_i)(a)) \cdot p(a)$$

$$\Big\| \; e_i(a)$$

$$\mathsf{pr}_2(k_i)(a)$$

Definition 5.2.2. A morphism $\varphi : \mathcal{A} \to \mathcal{B}$ of A-cocones is an *equivalence* if $\mathsf{pr}_1(A) \xrightarrow{\mathsf{pr}_1(\varphi)} \mathsf{pr}_1(B)$ is an equivalence.

Proposition 5.2.3. *For each diagram F in $A/\mathcal{U}$, the colimiting cocone under F is unique up to unique equivalence.*

Intuition

For all $i, j : \Gamma_0$ and $g : \Gamma_1(i, j)$, the commuting triangle of an A-cocone

$$\begin{array}{ccc} F_i & \xrightarrow{F_{i,j,g}} & F_j \\ & {\scriptstyle r_i}\searrow \qquad \swarrow{\scriptstyle r_j} & \\ & C & \end{array}$$

is equivalent to a homotopy $\eta_{i,j,g} : \mathsf{pr}_1(r_j) \circ \mathsf{pr}_1(F_{i,j,g}) \sim \mathsf{pr}_1(r_i)$ equipped with a commuting square

$$\begin{array}{ccc} \mathsf{pr}_1(r_j)(\mathsf{pr}_1(F_{i,j,g})(\mathsf{pr}_2(F_i)(a))) & \xRightarrow{\eta_{i,j,g}(\mathsf{pr}_2(F_i)(a))} & \mathsf{pr}_1(r_i)(\mathsf{pr}_2(F_i)(a)) \\ {\scriptstyle \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a))}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \mathsf{pr}_2(r_i)(a)} \\ \mathsf{pr}_1(r_j)(\mathsf{pr}_2(F_j)(a)) & \xRightarrow[\mathsf{pr}_2(r_j)(a)]{} & \mathsf{pr}_2(Y)(a) \end{array} \tag{2-c}$$

of paths for each $a : A$. It is this family of 2-cells which distinguishes the colimit of F, in $A/\mathcal{U}$, from $\mathsf{colim}_\Gamma(\mathcal{F}(F))$. The 2-cells affect $\mathsf{colim}_\Gamma(\mathcal{F}(F))$ by collapsing its nontrivial loops formed by paths of the form $\eta(\mathsf{pr}_2(F_i)(a))$. We call such loops *distinguished loops* in $\mathsf{colim}_\Gamma(\mathcal{F}(F))$. For example, if $i \equiv j$ and $F_{i,j,g} \equiv \mathsf{id}_{F_i}$, then (2-c) is equivalent to $\eta(\mathsf{pr}_2(F_i)(a)) = \mathsf{refl}_{\mathsf{pr}_1(r_i)(\mathsf{pr}_2(F_i)(a))}$. In this case, it fills the loop $\eta(\mathsf{pr}_2(F_i)(a))$.

5.3 Wedge sums in coslices

Let $A : \mathcal{U}$. Let Δ be a graph and G be an A-diagram over Δ. From this data, we can construct a diagram $\zeta(\Delta, G)$ over a graph Γ:

$$\begin{aligned} \Gamma_0 &:= \Delta_0 + \mathbf{1} & & \\ \Gamma_1(\mathsf{inl}(i), \mathsf{inl}(j)) &:= \Delta_1(i, j) & \qquad \zeta(\Delta, G)_{\mathsf{inl}(i)} &:= \mathsf{pr}_1(G_i) \\ \Gamma_1(\mathsf{inl}(i), \mathsf{inr}(*)) &:= \mathbf{0} & \qquad \zeta(\Delta, G)_{\mathsf{inr}(*)} &:= A \\ \Gamma_1(\mathsf{inr}(*), \mathsf{inl}(i)) &:= \mathbf{1} & \qquad \zeta(\Delta, G)_{\mathsf{inl}(i),\mathsf{inl}(j),g} &:= \mathsf{pr}_1(G_{i,j,g}) \\ \Gamma_1(\mathsf{inr}(*), \mathsf{inr}(*)) &:= \mathbf{0} & \qquad \zeta(\Delta, G)_{\mathsf{inr}(*),\mathsf{inl}(i),*} &:= \mathsf{pr}_2(G_i) \end{aligned}$$

If Δ is discrete, then the pushout

$$\begin{array}{ccc} \Delta_0 \times A & \xrightarrow{(i,a)\mapsto(i,\mathsf{pr}_2(G_i)(a))} & \sum_{i:\Delta_0}\mathsf{pr}_1(G_i) \\ {\scriptstyle \mathsf{pr}_2}\downarrow & & \downarrow{\scriptstyle\mathsf{inr}} \\ A & \xrightarrow[\mathsf{inl}]{} & D \end{array}$$

together with inl is the coproduct of the G_i in $A/\mathcal{U}$. We denote D by $\bigvee_{i:\Delta_0}\mathsf{pr}_1(G_i)$

Lemma 5.3.1. *Suppose that Δ is discrete. For each $i : \Delta_0$, the coproduct $\bigvee_{x:\Delta} G_x$ in $A/\mathcal{U}$ fits into a commuting diagram*

$$\begin{array}{ccc} & G_i & \\ {\scriptstyle (\mathsf{inr}(i,-),\lambda a.\,\mathsf{glue}(a,i)^{-1})}\swarrow & & \searrow{\scriptstyle (\iota_{\mathsf{inl}(i)},\kappa_{\zeta(\Delta,G)}(\mathsf{inr}(*),\mathsf{inl}(i)))} \\ \bigvee_{x:\Delta} G_x & \xrightarrow[\simeq_A]{} & \mathsf{colim}_{x:\Gamma}\,\zeta(\Delta,G,x) \end{array} \qquad (\texttt{tri-}\bigvee)$$

under A.

Proof. Define

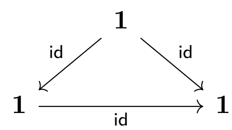
$$\begin{aligned}
(\varphi,\alpha) &: \left(\bigvee_{x:\Delta} G_x\right) \to \mathsf{colim}_{x:\Gamma}\,\zeta(\Delta,G,x) \\
\varphi(\mathsf{inl}(a)) &:\equiv \iota_{\mathsf{inr}(*)}(a) \\
\varphi(\mathsf{inr}(i,x)) &:\equiv \iota_{\mathsf{inl}(i)}(x) \\
\mathsf{ap}_\varphi(\mathsf{glue}(a,i)) &= \kappa_{\zeta(\Delta,G)}(\mathsf{inr}(*),\mathsf{inl}(i),a)^{-1}\cdot \mathsf{ap}_{\iota_{\mathsf{inl}(i)}}(\mathsf{pr}_2(G_{\mathsf{inr}(*),\mathsf{inl}(i),*})(a)) \\
\alpha &:\equiv \lambda a.\mathsf{refl}_{\iota_{\mathsf{inr}(*)}(a)}
\end{aligned}$$

Conversely, define

$$\begin{aligned}
\psi &: (\mathsf{colim}_{x:\Gamma}\,\zeta(\Delta,G,x)) \to \bigvee_{x:\Delta} G_x \\
\psi(\iota_{\mathsf{inr}(*)}(a)) &:\equiv \mathsf{inl}(a) \\
\psi(\iota_{\mathsf{inl}(i)}(x)) &:\equiv \mathsf{inr}(i,x) \\
\mathsf{ap}_\psi(\kappa_{\zeta(\Delta,G)}(\mathsf{inr}(*),\mathsf{inl}(i),a)) &= \mathsf{ap}_{\mathsf{inr}(i,-)}(\mathsf{pr}_2(G_{\mathsf{inr}(*),\mathsf{inl}(i),*})(a))\cdot \mathsf{glue}(a,i)^{-1}
\end{aligned}$$

It is easy to prove that φ and ψ are mutual inverses as ordinary functions. By Proposition 3.4.2, it follows that φ is an equivalence in $A/\mathcal{U}$. Moreover, it is easy to check that the triangle ($\texttt{tri-}\bigvee$) commutes in $A/\mathcal{U}$. ☐

Remark. It is *not* the case that $\mathsf{colim}^A_\Delta\,G$ and $\mathsf{colim}_{x:\Gamma}\,\zeta(\Delta,G,x)$ are equivalent in general. For example, the pointed colimit of the diagram $\mathbf{1} \xrightarrow{\mathsf{id}} \mathbf{1}$ is trivial, but the colimit of the augmented diagram



equals S^1. This situation may seem different from classical category theory, wherein colimits in coslice categories can be computed as colimits of augmented diagrams in the underlying category. Note, however, that the internal augmented diagram may add "composites" that are *not* interpreted as composites in the model of HoTT, but rather as unrelated arrows.

5.4 First construction of coslice colimits

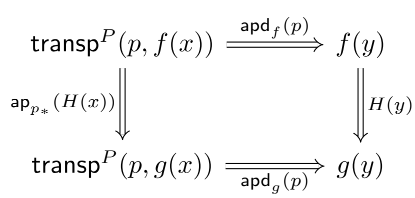
This section describes the main connection between ordinary colimits and coslice colimits. We build the colimit of F in a way that never creates an augmented diagram. We start with the ordinary colimit $\mathsf{colim}_\Gamma(\mathcal{F}(F))$ which ignores the coslice structure of F. Then, we glue onto this colimit the 2-cells required by the coslice colimit. We do this via a quotient of $\mathsf{colim}_\Gamma(\mathcal{F}(F))$ that fills its distinguished loops. For convenience, we begin by recording two variants of homotopy naturality.

Lemma 5.4.1. *Let X and Y be types and let $f, g : X \to Y$. Suppose that $H : f \sim g$. For all $x, y : X$ and $p : x = y$,*

$$\mathsf{ap}_g(p) \ = \ H(x)^{-1} \cdot \mathsf{ap}_f(p) \cdot H(y).$$

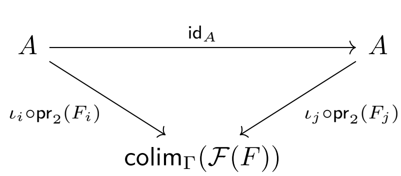
Proof. By path induction on p. ☐

Lemma 5.4.2. *Let X be a type and $P : X \to \mathcal{U}$. Let $f, g : \prod_{x:X} P(x)$. For all $x, y : X$, $p : x = y$, and $H : f \sim g$, we have a commuting square*



Proof. By path induction on p, ☐

Let $A : \mathcal{U}$. Consider a graph Γ and a diagram $F : \Gamma \to A/\mathcal{U}$ over Γ. Define $\psi : \mathsf{colim}_\Gamma A \to \mathsf{colim}_\Gamma(\mathcal{F}(F))$ as the function induced by the cocone



$$(a \mapsto \mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))$$

under the constant diagram at A. Then form the pushout square

$$
\begin{array}{ccc}
\mathsf{colim}_\Gamma A & \xrightarrow{\psi} & \mathsf{colim}_\Gamma(\mathcal{F}(F)) \\
\Big\downarrow{\scriptstyle [\mathsf{id}_A]_{i:\Gamma_0}} & & \Big\downarrow{\scriptstyle \mathsf{inr}} \\
A & \xrightarrow[\mathsf{inl}]{} & \mathcal{P}_A(F)
\end{array}
$$

We can form a A-cocone on $(\mathcal{P}_A(F), \mathsf{inl})$ under F

$$
\begin{array}{ccc}
\mathsf{pr}_1(F_i) & \xrightarrow{F^*_{i,j,g}} & \mathsf{pr}_1(F_j) \\
{\scriptstyle (\mathsf{inr}\circ\iota_i,\tau_i)}\searrow & {\scriptstyle \langle\delta_{i,j,g},\epsilon_{i,j,g}\rangle} & \swarrow{\scriptstyle (\mathsf{inr}\circ\iota_j,\tau_j)} \\
 & \mathcal{P}_A(F) &
\end{array}
\qquad (\tau_i(a) :\equiv \mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a))^{-1})
$$

as follows. We have a term $\delta_{i,j,g} :\equiv \lambda x.\, \mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(x)) \ : \ \mathsf{inr} \circ \iota_j \circ F_{i,j,g} \sim \mathsf{inr} \circ \iota_i$ Further, for each $a : A$, we have a chain $\epsilon_{i,j,g}(a)$ of equalities

$$
\begin{array}{c}
\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \tau_j(a) \\
\| {\scriptstyle \mathsf{Pl}(\mathsf{pr}_2(F_{i,j,g})(a),\tau_j(a))} \\
\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \tau_j(a) \cdot \mathsf{refl}_{\mathsf{inl}(a)} \\
\| {\scriptstyle \mathsf{ap}_{\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}\cdot\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1}\cdot\tau_j(a)\cdot -}(\mathsf{ap}_{\mathsf{ap}_{\mathsf{inl}}}(\rho_{[\mathsf{id}_A]}(i,j,g,a)))^{-1}} \\
\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \tau_j(a) \cdot \mathsf{ap}_{\mathsf{inl}}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a))) \\
\| {\scriptstyle \mathsf{ap}_{-\cdot\tau_j(a)\cdot\mathsf{ap}_{\mathsf{inl}}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a)))}(\mathsf{ap}_{\mathsf{ap}_{\mathsf{inr}}(-)^{-1}}(\rho_\psi(i,j,g,a)))^{-1}} \\
\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)))^{-1} \cdot \tau_j(a) \cdot \mathsf{ap}_{\mathsf{inl}}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a))) \\
\| {\scriptstyle \mathsf{Pl}(\kappa_{i,j,g}(a),\tau_j(a))} \\
(\kappa_{i,j,g}(a))_*(\tau_j(a)) \\
\| {\scriptstyle \mathsf{apd}_{\mathsf{glue}(-)^{-1}}(\kappa_{i,j,g}(a))} \\
\tau_i(a)
\end{array}
$$

It will be convenient to decompose $\epsilon_{i,j,g}(a)$ into the following chains of paths:

1. $E_1(i,j,g,a)$, the first path of $\epsilon_{i,j,g}(a)$
2. $E_2(i,j,g,a)$, the second path of $\epsilon_{i,j,g}(a)$
3. $E_3(i,j,g,a)$, the final three paths of $\epsilon_{i,j,g}(a)$.

Theorem 5.4.3. *Let* $(T, f_T) : A/\mathcal{U}$. *The* postcomp *function*

$$
\begin{aligned}
e_{F,T} \ &: \ ((\mathcal{P}_A(F), \mathsf{inl}) \to_A (T, f_T)) \to \mathsf{Cocone}_F(T, f_T) \\
e_{F,T}(f, f_p) \ &\coloneqq \\
&\Big(\lambda i.\, \big(f \circ \mathsf{inr} \circ \iota_i, \lambda a.\, \mathsf{ap}_f(\tau_i(a)) \cdot f_p(a)\big), \lambda i \lambda j \lambda g.\, \Big(\lambda x.\, \mathsf{ap}_f(\delta_{i,j,g}(x)), \lambda a. \Theta_{\delta_{i,j,g}}(f^*, a) \cdot \mathsf{ap}_{-\cdot f_p(a)}(\mathsf{ap}_{\mathsf{ap}_f}(\epsilon_{i,j,g}(a)))\Big)\Big)
\end{aligned}
$$

is an equivalence.[2]

Proof. We define a quasi-inverse of $e_{F,T}$ as follows. Consider a cocone $(r, K) : \mathsf{Cocone}_F(T, f_T)$ under F on (T, f_T). For all $i : \Gamma_0$ and $a : A$, we have that

$$
\begin{aligned}
& f_T(a) \\
= \ & \mathsf{pr}_1(r_i)(\mathsf{pr}_2(F_i(a))) && (\mathsf{pr}_2(r_i)(a)^{-1}) \\
\equiv \ & \mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r, K))(\mathsf{pr}_2(F_i(a)))
\end{aligned}
$$

and for all $i, j : \Gamma_0$, $g : \Gamma_1(i, j)$, and $a : A$, we have a chain $\eta_{i,j,g}(a)$ of equalities

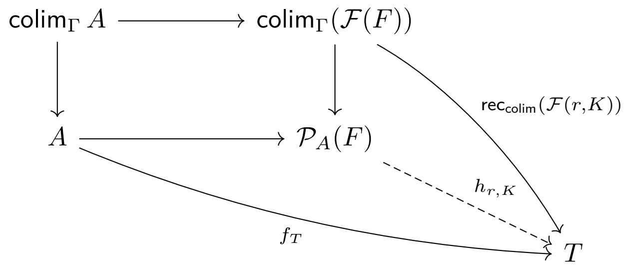
$$
\begin{array}{c}
\mathsf{transp}^{x \mapsto f_T([\mathsf{id}_A](x)) = \mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))(\psi(x))}(\kappa_{i,j,g}(a), \mathsf{pr}_2(r_j)(a)^{-1}) \\
\Vert \\
\scriptstyle \mathsf{PI}(\kappa_{i,j,g}(a), \mathsf{pr}_2(r_j)(a)) \\
\Vert \\
\mathsf{ap}_{f_T}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a)))^{-1} \cdot \mathsf{pr}_2(r_j)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))) \\
\Vert \\
\scriptstyle \mathsf{ap}_{\mathsf{ap}_{f_T}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a)))^{-1} \cdot \mathsf{pr}_2(r_j)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))}(-)}(\rho_\psi(i,j,g,a)) \\
\Vert \\
\mathsf{ap}_{f_T}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a)))^{-1} \cdot \mathsf{pr}_2(r_j)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \\
\Vert \\
\scriptstyle \mathsf{PI}(\mathsf{pr}_2(F_{i,j,g})(a)) \\
\Vert \\
\mathsf{ap}_{f_T}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a)))^{-1} \cdot \mathsf{pr}_2(r_j)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \\
\Vert \\
\scriptstyle \mathsf{ap}_{\mathsf{ap}_{f_T}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a)))^{-1} \cdot \mathsf{pr}_2(r_j)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \_}(\rho_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))}(i,j,g,\mathsf{pr}_2(F_i)(a))) \\
\Vert \\
\mathsf{ap}_{f_T}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a)))^{-1} \cdot \mathsf{pr}_2(r_j)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \mathsf{pr}_1(K_{i,j,g})(\mathsf{pr}_2(F_i)(a)) \\
\Vert \\
\scriptstyle \mathsf{PI}(\mathsf{pr}_2(r_j)(a), \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)), \mathsf{pr}_1(K_{i,j,g})(\mathsf{pr}_2(F_i)(a))) \\
\Vert \\
\mathsf{ap}_{f_T}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a)))^{-1} \cdot \Big(\mathsf{pr}_1(K_{i,j,g})(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)\Big)^{-1} \\
\Vert \\
\scriptstyle \mathsf{ap}_{-\cdot \left(\mathsf{pr}_1(K_{i,j,g})(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)\right)^{-1}}(\mathsf{ap}_{\mathsf{ap}_{f_T}(-)^{-1}}(\rho_{[\mathsf{id}_A]}(i,j,g,a))) \\
\Vert \\
\Big(\mathsf{pr}_1(K_{i,j,g})(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)\Big)^{-1} \\
\Vert \\
\scriptstyle \mathsf{ap}_{\_^{-1}}(\mathsf{pr}_2(K_{i,j,g})(a)) \\
\Vert \\
\mathsf{pr}_2(r_i)(a)^{-1}
\end{array}
$$

[2] This is formalized in [7, `Colimit-code/Main-Theorem/CosColim-Iso.agda`].

This gives us a function

$$\sigma \;:\; \prod_{x:\mathsf{colim}_\Gamma A} f_T([\mathsf{id}_A](x)) = \mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))(\psi(x)) \tag{†}$$

and thus a function $h_{r,K} : \mathcal{P}_A(F) \to T$



defined by induction on $\mathcal{P}_A(F)$. Since $h(\mathsf{inl}(a)) \equiv f_T(a)$, we have a term

$$\left(h_{r,K}, \lambda a.\mathsf{refl}_{f_T(a)}\right) \;:\; \mathcal{P}_A(F) \to_A T$$

Observe that

$$\begin{aligned} & e_{F,T}(h_{r,K}, \lambda a.\mathsf{refl}_{f_T(a)}) \\ \equiv\; & \left(\lambda i.\left(h_{r,K} \circ \mathsf{inr} \circ \iota_i, \mathsf{ap}_{h_{r,K}}(\tau_i(a)) \cdot \mathsf{refl}_{f_T(a)}\right), \lambda i \lambda j \lambda g.\left(\lambda x.\, \mathsf{ap}_{h_{r,K}}(\delta_{i,j,g}(x)), \lambda a.\Theta_{\delta_{i,j,g}}(h^*_{r,K}, a) \cdot \mathsf{ap}_{\mathsf{ap}_{h_{r,K}}(-)\cdot \mathsf{refl}_{f_T(a)}}(\epsilon_{i,j,g}(a))\right)\right) \end{aligned}$$

and that $h_{r,K} \circ \mathsf{inr} \circ \iota_i \equiv \mathsf{pr}_1(r_i)$. For each $i : \Gamma_0$ and $a : A$, we have a chain $P_i(a)$ of equalities

$$\begin{aligned} & \mathsf{ap}_{h_{r,K}}(\tau_i(a)) \cdot \mathsf{refl}_{f_T(a)} \\ =\; & \mathsf{ap}_{h_{r,K}}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1} && (\mathsf{PI}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))) \\ =\; & \left(\mathsf{pr}_2(r_i)(a)^{-1}\right)^{-1} && (\mathsf{ap}_{-^{-1}}(\rho_{h_{r,K}}(\iota_i(a)))) \\ =\; & \mathsf{pr}_2(r_i)(a). && (\mathsf{PI}(\mathsf{pr}_2(r_i)(a))) \end{aligned}$$

Notation. We denote the path $\mathsf{PI}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))$ by $\Delta_i(a)$.

Moreover, for all $i, j : \Gamma_0$, $g : \Gamma_1(i,j)$, and $x : \mathsf{pr}_1(F_i)$, we have a chain $Q_{i,j,g}(x)$

$$\begin{aligned} & \mathsf{ap}_{h_{r,K}}(\delta_{i,j,g}(x)) \cdot \mathsf{refl}_{h_{r,K}(\mathsf{inr}(\iota_i(x)))} \\ \equiv\; & \mathsf{ap}_{h_{r,K}}(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(x))) \cdot \mathsf{refl}_{h_{r,K}(\mathsf{inr}(\iota_i(x)))} \\ =\; & \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))}(\kappa_{i,j,g}(x)) && (\mathsf{PI}(\kappa_{i,j,g}(x))) \\ =\; & \mathsf{pr}_1(K_{i,j,g})(x). && (\rho_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))}(i,j,g,x)) \end{aligned}$$

By Lemma 3.5.2, we must prove that for all $i, j : \Gamma_0$, $g : \Gamma_1(i,j)$, and $a : A$,

$$\begin{array}{c}
\mathsf{ap}_{_^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)}(Q_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \\
\Xi(P, \left(\mathsf{ap}_{h_{r,K}}(\delta_{i,j,g}(x)), \Theta_{\delta_{i,j,g}}(h^*_{r,K}, a) \cdot \mathsf{ap}_{\mathsf{ap}_{h_{r,K}}(-) \cdot \mathsf{refl}_{f_T(a)}}(\epsilon_{i,j,g}(a)) \right), a) \\
\| \\
\mathsf{pr}_2(K_{i,j,g})
\end{array}$$

To this end, note that

$$\begin{array}{c}
\Delta_i(a) \cdot \mathsf{ap}_{_^{-1}}(\rho_{h_{r,K}}(\iota_i(a))) \\
\| \\
\mathsf{transp}^{x \mapsto \mathsf{ap}_{h_{r,K}}(\mathsf{glue}_{\mathcal{P}_A(F)}(x)^{-1}) \cdot \mathsf{refl}_{f_T([\mathsf{id}_A](x))} = \sigma(x)^{-1}}(\kappa_{i,j,g}(a), \Delta_j(a) \cdot \mathsf{ap}_{_^{-1}}(\rho_{h_{r,K}}(\iota_j(a)))) \\
\| \\
\mathsf{apd}_{\mathsf{ap}_{h_{r,K}}(\mathsf{glue}_{\mathcal{P}_A(F)}(-)^{-1}) \cdot \mathsf{refl}_{f_T([\mathsf{id}_A](-))}}(\kappa_{i,j,g}(a))^{-1} \cdot \\
\mathsf{ap}_{\mathsf{transp}^{x \mapsto \mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))(\psi(x)) = f_T([\mathsf{id}_A](x))}(\kappa_{i,j,g}(a), -)}(\Delta_j(a) \cdot \mathsf{ap}_{_^{-1}}(\rho_{h_{r,K}}(\iota_j(a)))) \cdot \mathsf{apd}_{\sigma(-)^{-1}}(\kappa_{i,j,g}(a))
\end{array}$$

and that the triangle

$$\begin{array}{ccc}
\mathsf{transp}^{x \mapsto \mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))(\psi(x)) = f_T([\mathsf{id}_A](x))}(\kappa_{i,j,g}(a), \left(\mathsf{pr}_2(r_j)(a)^{-1}\right)^{-1}) & \xRightarrow{\mathsf{apd}_{\sigma(-)^{-1}}(\kappa_{i,j,g}(a))} & \left(\mathsf{pr}_2(r_i)(a)^{-1}\right)^{-1} \\
\Big\Downarrow {\scriptstyle \mathsf{Pl}(\kappa_{i,j,g}(a))} & \nearrow {\scriptstyle \mathsf{ap}_{_^{-1}}(\mathsf{apd}_\sigma(\kappa_{i,j,g}(a)))} & \\
\mathsf{transp}^{x \mapsto f_T([\mathsf{id}_A](x)) = \mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))(\psi(x))}(\kappa_{i,j,g}(a), \mathsf{pr}_2(r_j)(a)^{-1})^{-1} & &
\end{array}$$

commutes, where $\mathsf{apd}_\sigma(\kappa_{i,j,g}(a)) = \eta_{i,j,g}(a)$ by the induction principle for σ. Therefore, after unfolding Ξ, we want to show that for each $a : A$, $\mathsf{pr}_2(K_{i,j,g})(a)$ equals the chain $C_\Xi(a)$, shown on the next page.

$$\mathsf{pr}_1(K_{i,j,g})(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)$$

$$\Vert$$

$$\mathsf{ap}_{-^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)}(Q_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1}$$

$$\Vert$$

$$\left(\mathsf{ap}_{h_{r,K}}(\delta_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \cdot \mathsf{refl}_{h_{r,K}(\mathsf{inr}(\iota_i(\mathsf{pr}_2(F_i)(a))))}\right)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)$$

$$\Vert$$

$$\mathsf{PI}(\mathsf{pr}_2(F_{i,j,g})(a))$$

$$\Vert$$

$$\left(\left(\mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}\right) \cdot \mathsf{ap}_{h_{r,K}}(\delta_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \cdot \mathsf{refl}_{h_{r,K}(\mathsf{inr}(\iota_i(\mathsf{pr}_2(F_i)(a))))}\right)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)$$

$$\Vert$$

$$\mathsf{ap}_{\left(\left(\mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot - \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}\right) \cdot \mathsf{ap}_{h_{r,K}}(\delta_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \cdot \mathsf{refl}_{h_{r,K}(\mathsf{inr}(\iota_i(\mathsf{pr}_2(F_i)(a))))}\right)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)}(\widetilde{P}_j(a))$$

$$\Vert$$

$$\left(\left(\mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \left(\mathsf{pr}_2(r_j)(a) \cdot \left(\mathsf{ap}_{h_{r,K}}(\tau_j(a)) \cdot \mathsf{refl}_{f_T(a)}\right)^{-1}\right) \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}\right) \cdot \mathsf{ap}_{h_{r,K}}(\delta_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \cdot \mathsf{refl}_{h_{r,K}(\mathsf{inr}(\iota_i(\mathsf{pr}_2(F_i)(a))))}\right)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(r_j)(a)$$

$$\Vert$$

$$\mathsf{PI}(\mathsf{pr}_2(F_{i,j,g})(a), \mathsf{pr}_2(r_j)(a), \mathsf{ap}_{h_{r,K}}(\tau_j(a)) \cdot \mathsf{refl}_{f_T(a)}, \mathsf{ap}_{h_{r,K}}(\delta_{i,j,g}(\mathsf{pr}_2(F_i)(a))))$$

$$\Vert$$

$$\mathsf{ap}_{h_{r,K}}(\delta_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(r_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{h_{r,K}}(\tau_j(a)) \cdot \mathsf{refl}_{f_T(a)}$$

$$\Vert$$

$$\Theta_{\delta_{i,j,g}}(h^*_{r,K}, a) \cdot \mathsf{ap}_{\mathsf{ap}_{h_{r,K}}(-) \cdot \mathsf{refl}_{f_T(a)}}(\epsilon_{i,j,g}(a))$$

$$\Vert$$

$$\mathsf{ap}_{h_{r,K}}(\tau_i(a)) \cdot \mathsf{refl}_{f_T(a)}$$

$$\Vert$$

$$\mathsf{apd}_{\mathsf{ap}_{h_{r,K}}(\mathsf{glue}_{\mathcal{P}_A(F)}(-)^{-1}) \cdot \mathsf{refl}_{f_T([\mathsf{id}_A](-))}}(\kappa_{i,j,g}(a))^{-1}$$

$$\Vert$$

$$\mathsf{transp}^{x \mapsto \mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))(\psi(x)) = f_T([\mathsf{id}_A](x))}(\kappa_{i,j,g}(a), \mathsf{ap}_{h_{r,K}}(\tau_j(a)) \cdot \mathsf{refl}_{f_T(a)})$$

$$\Vert$$

$$\mathsf{ap}_{\mathsf{transp}^{x \mapsto \mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(r,K))(\psi(x)) = f_T([\mathsf{id}_A](x))}(\kappa_{i,j,g}(a), -)}(\Delta_j(a) \cdot \mathsf{ap}_{-^{-1}}(\rho_{h_{r,K}}(\iota_j(a))))$$

$$\Vert$$

$$(\kappa_{i,j,g}(a))_*\left(\left(\mathsf{pr}_2(r_j)(a)^{-1}\right)^{-1}\right)$$

$$\Vert$$

$$\mathsf{PI}(\kappa_{i,j,g}(a))$$

$$\Vert$$

$$(\kappa_{i,j,g}(a))_*(\mathsf{pr}_2(r_j)(a)^{-1})^{-1}$$

$$\Vert$$

$$\mathsf{ap}_{-^{-1}}(\eta_{i,j,g}(a))$$

$$\Vert$$

$$\left(\mathsf{pr}_2(r_i)(a)^{-1}\right)^{-1}$$

$$\Vert$$

$$\mathsf{PI}(\mathsf{pr}_2(r_i)(a))$$

$$\Vert$$

$$\mathsf{pr}_2(r_i)(a)$$

We can reduce $C_{\Xi}(a)$ to $\mathsf{pr}_2(K_{i,j,g})(a)$, which appears in $\eta_{i,j,g}(a)$, in a bottom-up fashion. This process iteratively removes the ρ terms appearing in $C_{\Xi}(a)$. We refer the reader to the Agda formalization for the full reduction (see the [7, `Colimit-code/R-L-R`] folder).

So far, we've defined a right inverse of $e_{F,T}$. We next want to prove that this is also a left inverse.[3] To this end, suppose that $(f, f_p) : (\mathcal{P}_A(F) \to_A T)$ and let $E_1 :\equiv \mathsf{pr}_1(e_{F,T}(f, f_p))$ and $E_2 :\equiv \mathsf{pr}_2(e_{F,T}(f, f_p))$. We want to find terms

$$\begin{aligned} \alpha &: \prod_{x:\mathcal{P}_A(F)} f(x) = \overbrace{h_{E_1,E_2}}^{\tilde{h}}(x) \\ \widehat{\alpha} &: \prod_{a:A} \alpha(\mathsf{inl}(a))^{-1} \cdot f_p(a) = \mathsf{refl}_{f_T(a)} \end{aligned}$$

To construct α, we use Lemma 5.1.5. For each $a : A$, we have that

$$\begin{aligned} & f(\mathsf{inl}(a)) \\ =\ & f_T(a) && (f_p(a)) \\ \equiv\ & \tilde{h}(\mathsf{inl}(a)). \end{aligned}$$

Already, we see that once α is constructed, it will be easy to derive $\widehat{\alpha}$ from it. Moreover,

$$f(\mathsf{inr}(\iota_i(x))) \equiv \tilde{h}(\mathsf{inr}(\iota_i(x))),$$

We also have a chain $V_{i,j,g}(x)$ of equalities

$$\begin{array}{c} \mathsf{transp}^{y \mapsto f(\mathsf{inr}(y)) = \tilde{h}(\mathsf{inr}(y))}(\kappa_{i,j,g}(x), \mathsf{refl}_{f(\mathsf{inr}(\iota_j(F_{i,j,g}(x))))}) \\ \Big\| \ \mathsf{PI}(\kappa_{i,j,g}(x)) \\ \mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(x)))^{-1} \cdot \mathsf{ap}_{\underbrace{\tilde{h} \circ \mathsf{inr}}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}}(\kappa_{i,j,g}(x)) \\ \Big\| \ \mathsf{ap}_{\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(x)))^{-1} \cdot \_}(\rho_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}(i,j,g,x)) \\ \mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(x)))^{-1} \cdot \mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(x))) \\ \Big\| \ \mathsf{PI}(\kappa_{i,j,g}(x)) \\ \mathsf{refl}_{f(\mathsf{inr}(\iota_i(x)))} \end{array}$$

By induction on $\mathsf{colim}_{\Gamma}(\mathcal{F}(F))$, this gives us a term

$$\gamma : \prod_{x:\mathsf{colim}_{\Gamma}(\mathcal{F}(F))} f(\mathsf{inr}(x)) = \tilde{h}(\mathsf{inr}(x)).$$

[3]This is formalized in [7, `Colimit-code/L-R-L`].

For all $i : \Gamma_0$ and $a : A$, we have the chain $R_i(a)$ of equalities

$$
\begin{aligned}
& \Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1} \cdot f_p(a)\Big) \cdot \mathsf{ap}_{\tilde{h}}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a))) \\
= \; & \Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1} \cdot f_p(a)\Big) \cdot \Big(\mathsf{ap}_f(\tau_i(a)) \cdot f_p(a)\Big)^{-1} && \Big(\mathsf{ap}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1}\cdot f_p(a)\right)\cdot -}(\rho_{\tilde{h}}(\iota_i(a)))\Big) \\
= \; & \mathsf{refl}_{f(\mathsf{inr}(\iota_i(\mathsf{pr}_2(F_i)(a))))} && \Big(\mathsf{PI}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)), f_p(a))\Big) \\
\equiv \; & \gamma(\psi(\iota_i(a))).
\end{aligned}
$$

Further, for all $i, j : \Gamma_0$, $g : \Gamma_1(i,j)$, and $a : A$,

$$
\begin{array}{c}
\mathsf{transp}^{x \mapsto \left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(x))^{-1}\cdot f_p([\mathsf{id}_A](x))\right)\cdot \mathsf{ap}_{\tilde{h}}(\mathsf{glue}_{\mathcal{P}_A(F)}(x)) =_{f(\mathsf{inr}(\psi(x)))=\tilde{h}(\mathsf{inr}(\psi(x)))} \gamma(\psi(x))}(\kappa_{i,j,g}(a), R_j(a)) \\
\Vert \\
\mathsf{apd}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(-))^{-1}\cdot f_p([\mathsf{id}_A](-))\right)\cdot \mathsf{ap}_{\tilde{h}}(\mathsf{glue}_{\mathcal{P}_A(F)}(-))}(\kappa_{i,j,g}(a))^{-1} \cdot \\
\mathsf{ap}_{\mathsf{transp}^{x\mapsto f(\mathsf{inr}(\psi(x)))=\tilde{h}(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a),-)}(R_j(a)) \cdot \mathsf{apd}_{\gamma(\psi(-))}(\kappa_{i,j,g}(a))
\end{array}
$$

We must prove that this chain of equalities equals $R_i(a)$. By Lemma 5.4.2, we have a commuting square

$$
\begin{array}{ccc}
(\kappa_{i,j,g}(a))_* \left(\Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))^{-1} \cdot f_p(a)\Big) \cdot \mathsf{ap}_{\tilde{h}}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))\right) & \xRightarrow{\mathsf{apd}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(-))^{-1}\cdot f_p([\mathsf{id}_A](-))\right)\cdot \mathsf{ap}_{\tilde{h}}(\mathsf{glue}_{\mathcal{P}_A(F)}(-))}(\kappa_{i,j,g}(a))} & \Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1} \cdot f_p(a)\Big) \cdot \mathsf{ap}_{\tilde{h}}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a))) \\
\Big\Downarrow \; \mathsf{ap}_{\mathsf{transp}^{x\mapsto f(\mathsf{inr}(\psi(x)))=\tilde{h}(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a),-)}\Big(\mathsf{ap}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))^{-1}\cdot f_p(a)\right)\cdot -}(\rho_{\tilde{h}}(\iota_j(a)))\Big) & & \Big\Uparrow \; \mathsf{ap}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1}\cdot f_p(a)\right)\cdot -}(\rho_{\tilde{h}}(\iota_i(a))) \\
(\kappa_{i,j,g}(a))_* \left(\Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))^{-1} \cdot f_p(a)\Big) \cdot \Big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\Big)^{-1}\right) & \xRightarrow[\mathsf{apd}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(-))^{-1}\cdot f_p([\mathsf{id}_A](-))\right)\cdot \sigma(-)}(\kappa_{i,j,g}(a))]{} & \Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1} \cdot f_p(a)\Big) \cdot \Big(\mathsf{ap}_f(\tau_i(a)) \cdot f_p(a)\Big)^{-1}
\end{array}
$$

Therefore, it suffices to show that

$$
\begin{array}{c}
\mathsf{apd}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(-))^{-1}\cdot f_p([\mathsf{id}_A](-))\right)\cdot \sigma(-)}(\kappa_{i,j,g}(a))^{-1} \cdot \\
\mathsf{ap}_{\mathsf{transp}^{x\mapsto f(\mathsf{inr}(\psi(x)))=\tilde{h}(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a),-)}(\mathsf{PI}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)), f_p(a))) \cdot \mathsf{apd}_{\gamma(\psi(-))}(\kappa_{i,j,g}(a)) \\
\Vert \\
\mathsf{PI}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)), f_p(a))
\end{array}
$$

where the two PI terms refer to those in $R_j(a)$ and $R_i(a)$, respectively.

We have two commuting triangles of paths, shown on the next page.

$$
\begin{array}{c}
(\kappa_{i,j,g}(a))_*\left(\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))^{-1}\cdot f_p(a)\right)\cdot\sigma(\iota_j(a))\right) \\
\swarrow\!\!\!\!\!\!\Leftarrow \; \mathsf{Pl}(\kappa_{i,j,g}(a)) \qquad\qquad \mathsf{apd}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(-))^{-1}\cdot f_p([\mathsf{id}_A](-))\right)\cdot\sigma(-)}(\kappa_{i,j,g}(a)) \; \Rightarrow\!\!\!\!\searrow \\
\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1}\cdot f_p(a)\right)\cdot(\kappa_{i,j,g}(a))_*(\sigma(\iota_j(a))) \xRightarrow[\mathsf{ap}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1}\cdot f_p(a)\right)\cdot -}(\mathsf{apd}_\sigma(\kappa_{i,j,g}(a)))]{} \left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1}\cdot f_p(a)\right)\cdot\sigma(\iota_i(a))
\end{array}
$$

$$
\begin{array}{c}
(\kappa_{i,j,g}(a))_*(\gamma(\psi(\iota_j(a)))) \\
\swarrow\!\!\!\!\!\!\Leftarrow \; \mathsf{Pl}(\kappa_{i,j,g}(a)) \qquad\qquad \mathsf{apd}_{\gamma(\psi(-))}(\kappa_{i,j,g}(a)) \; \Rightarrow\!\!\!\!\searrow \\
\mathsf{ap}_\psi(\kappa_{i,j,g}(a))_*(\gamma(\psi(\iota_j(a)))) \xRightarrow[\mathsf{apd}_\gamma(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)))]{} \gamma(\psi(\iota_i(a)))
\end{array}
$$

Note that $\mathsf{apd}_{\sigma}(\kappa_{i,j,g}(a)) = \eta_{i,j,g}(a)$ by the induction principle for $\mathsf{colim}_{\Gamma} A$. Further,

$$\mathsf{apd}_{\gamma}(\mathsf{ap}_{\psi}(\kappa_{i,j,g}(a)))$$
$$\|$$
$$\mathsf{ap}_{-_*(\underbrace{\gamma(\psi(\iota_j(a)))}_{\mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))}})}(\rho_{\psi}(i,j,g,a)) \cdot \mathsf{apd}_{\gamma}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))$$
$$\|$$
$$\mathsf{ap}_{-_*(\mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))})}(\rho_{\psi}(i,j,g,a)) \cdot \mathsf{Pl}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{apd}_{\gamma}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))$$

where $\mathsf{Pl}(\mathsf{pr}_2(F_{i,j,g})(a))$ has type

$$\Big(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))\Big)_*(\mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))})$$
$$\|$$
$$(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))_*(\gamma(\iota_j(F_{i,j,g}(\mathsf{pr}_2(F_i)(a)))))$$

Note that $\mathsf{apd}_{\gamma}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))) = V_{i,j,g}(\mathsf{pr}_2(F_i)(a))$ by the induction principle for $\mathsf{colim}_{\Gamma}(\mathcal{F}(F))$.

Now, let

$$Y_{i,j,g}(a) \coloneqq \Theta_{\lambda x.\, \mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(x))}(f^*, a) \cdot \mathsf{ap}_{-\cdot f_p(a)}(\mathsf{ap}_{\mathsf{ap}_f}(\epsilon_{i,j,g}(a)))$$

For each $s : f(\mathsf{inr}(\psi(\iota_j(a)))) = \tilde{h}(\mathsf{inr}(\psi(\iota_j(a))))$, consider the chain $\chi(s)$ of equalities

$$\mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x))) = \tilde{h}(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a), s)$$
$$\Big\| \; \mathsf{Pl}(\kappa_{i,j,g}(a), s)$$
$$\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\psi}(\kappa_{i,j,g}(a))))^{-1} \cdot s \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}(\mathsf{ap}_{\psi}(\kappa_{i,j,g}(a)))$$
$$\Big\| \; \mathsf{ap}_{\cdots\cdot\_}(\mathsf{ap}_{\mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}}(\rho_{\psi}(i,j,g,a)))$$
$$\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\psi}(\kappa_{i,j,g}(a))))^{-1} \cdot s \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))$$
$$\Big\| \; \mathsf{ap}_{\cdots\cdot\_}(\mu_1(i,j,g,a))$$
$$\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\psi}(\kappa_{i,j,g}(a))))^{-1} \cdot s \cdot \Big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\Big) \cdot \Big(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \mathsf{ap}_{f \circ \mathsf{inr} \circ \iota_j}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\Big)^{-1}$$
$$\Big\| \; \mathsf{ap}_{\cdots\cdot\_}(\mathsf{ap}_{(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)) \cdot \_^{-1}}(Y_{i,j,g}(a)))$$
$$\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\psi}(\kappa_{i,j,g}(a))))^{-1} \cdot s \cdot \Big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\Big) \cdot \Big(\mathsf{ap}_f(\tau_i(a)) \cdot f_p(a)\Big)^{-1}$$
$$\Big\| \; \mathsf{ap}_{\cdots\cdot\_\cdot(\mathsf{ap}_f(\tau_i(a)) \cdot f_p(a))^{-1}}(\mu_2(i,j,g,a))$$
$$\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\psi}(\kappa_{i,j,g}(a))))^{-1} \cdot s \cdot \Big(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\psi}(\kappa_{i,j,g}(a)))) \cdot \Big(\mathsf{ap}_f(\tau_i(a)) \cdot f_p(a)\Big) \cdot \mathsf{refl}_{f_T(a)}\Big) \cdot \Big(\mathsf{ap}_f(\tau_i(a)) \cdot f_p(a)\Big)^{-1}$$
$$\Big\| \; \mathsf{Pl}(\tau_i(a), f_p(a), \kappa_{i,j,g}(a))$$
$$\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\psi}(\kappa_{i,j,g}(a))))^{-1} \cdot s \cdot \mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\psi}(\kappa_{i,j,g}(a))))$$
$$\Big\| \; \mathsf{Pl}(\kappa_{i,j,g}(a), s)$$
$$\mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x))) = f(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a), s)$$

where $\mu_1(i,j,g,a)$ and $\mu_2(i,j,g,a)$ denote the chains of equalities

$$
\begin{array}{c}
\mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \\
\Big\| \\
\mathsf{Pl}(\mathsf{pr}_2(F_{i,j,g})(a)) \\
\Big\| \\
\mathsf{ap}_{\underbrace{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2)) \circ \iota_j}_{f \circ \mathsf{inr} \circ \iota_j}}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \\
\Big\| \\
\mathsf{ap}_{\mathsf{ap}_{f \circ \mathsf{inr} \circ \iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \_}(\rho_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}(i,j,g,\mathsf{pr}_2(F_i)(a))) \\
\Big\| \\
\mathsf{ap}_{f \circ \mathsf{inr} \circ \iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))) \\
\Big\| \\
\mathsf{Pl}(\tau_j(a), f_p(a)) \\
\Big\| \\
\big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big) \cdot \big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big)^{-1} \cdot \mathsf{ap}_{f \circ \mathsf{inr} \circ \iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))) \\
\Big\| \\
\mathsf{ap}_{\left(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\right) \cdot \_}(\mathsf{Pl}(\tau_j(a), f_p(a), \mathsf{pr}_2(F_{i,j,g})(a), \mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))) \\
\Big\| \\
\big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big) \cdot \Big(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \mathsf{ap}_{f \circ \mathsf{inr} \circ \iota_j}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\Big)^{-1}
\end{array}
$$

$$
\begin{array}{c}
\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a) \\
\Big\| \\
\mathsf{apd}_{\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(-)^{-1}) \cdot f_p([\mathsf{id}_A](-))}(\kappa_{i,j,g}(a)^{-1})^{-1} \\
\Big\| \\
\mathsf{transp}^{y \mapsto f(\mathsf{inr}(\psi(y))) = f_T([\mathsf{id}_A](y))}(\kappa_{i,j,g}(a)^{-1}, \mathsf{ap}_f(\tau_i) \cdot f_p(a)) \\
\Big\| \\
\mathsf{Pl}(\kappa_{i,j,g}(a), \tau_i(a), f_p(a)) \\
\Big\| \\
\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)))) \cdot \big(\mathsf{ap}_f(\tau_i) \cdot f_p(a)\big) \cdot \mathsf{ap}_{f_T}(\mathsf{ap}_{[\mathsf{id}_A]}(\kappa_{i,j,g}(a)))^{-1} \\
\Big\| \\
\mathsf{ap}_{\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)))) \cdot \left(\mathsf{ap}_f(\tau_i) \cdot f_p(a)\right) \cdot \mathsf{ap}_{f_T}(-)^{-1}}(\rho_{[\mathsf{id}_A]}(i,j,g,a)) \\
\Big\| \\
\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)))) \cdot \big(\mathsf{ap}_f(\tau_i(a)) \cdot f_p(a)\big) \cdot \mathsf{refl}_{f_T(a)}
\end{array}
$$

respectively. By Lemma 5.4.1, we have a path

$$
\begin{array}{c}
\mathsf{ap}_{\mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x))) = \tilde{h}(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a), -)}(\mathsf{Pl}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)), f_p(a))) \\
\Big\| \\
\chi(\big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))^{-1} \cdot f_p(a)\big) \cdot \big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big)^{-1}) \cdot \\
\quad \mathsf{ap}_{\mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x))) = f(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a), -)}(\mathsf{Pl}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)), f_p(a))) \cdot \chi(\mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))})^{-1}
\end{array}
$$

where $\mathsf{PI}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)), f_p(a))$ has type

$$\Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))^{-1} \cdot f_p(a)\Big) \cdot \big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big)^{-1} \;=\; \mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))}$$

We also have a commuting square

$$\begin{array}{ccc}
\Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1} \cdot f_p(a)\Big) \cdot \big(\mathsf{ap}_f(\tau_i(a)) \cdot f_p(a)\big)^{-1} & \xRightarrow{\mathsf{PI}(\kappa_{i,j,g}(a))} & \mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x)))=f(\mathsf{inr}(\psi(x)))}\Big(\kappa_{i,j,g}(a), \Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))^{-1} \cdot f_p(a)\Big) \cdot \big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big)^{-1}\Big) \\
\Big\Downarrow \; \mathsf{ap}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1}\cdot f_p(a)\right)\cdot -}(\eta_{i,j,g}(a))^{-1} & & \Big\Uparrow \; \chi\Big(\Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))^{-1}\cdot f_p(a)\Big)\cdot\big(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\big)^{-1}\Big) \\
\Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)))^{-1}\cdot f_p(a)\Big)\cdot \mathsf{transp}^{x \mapsto f_T([\mathsf{id}_A](x))=\tilde{h}(\mathsf{inr}(\psi(x)))}\Big(\kappa_{i,j,g}(a), \big(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\big)^{-1}\Big) & \xRightarrow[\mathsf{PI}(\kappa_{i,j,g}(a))]{} & \mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x)))=\tilde{h}(\mathsf{inr}(\psi(x)))}\Big(\kappa_{i,j,g}(a), \Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))^{-1} \cdot f_p(a)\Big) \cdot \big(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\big)^{-1}\Big)
\end{array}$$

We've now put

$$\mathsf{apd}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(-))^{-1}\cdot f_p([\mathsf{id}_A](-))\right)\cdot\sigma(-)}(\kappa_{i,j,g}(a))^{-1}\cdot\chi(\Big(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)))^{-1}\cdot f_p(a)\Big)\cdot\big(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\big)^{-1})$$

into a useful form.

Let's put $\chi(\mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))})^{-1}\cdot\mathsf{apd}_{\gamma(\psi(-))}(\kappa_{i,j,g}(a))$ into a useful form as well. Consider the following three chains of equalities:

$$\begin{aligned}
&\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1}\cdot\big(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))\cdot\big(\mathsf{ap}_f(\tau_i(a))\cdot f_p(a)\big)\cdot\mathsf{refl}_{f_T(a)}\big)\cdot\big(\mathsf{ap}_f(\tau_i(a))\cdot f_p(a)\big)^{-1}\\
&\quad\Big\Vert\ \mathsf{ap}_{\cdots\cdot-\cdot\left(\mathsf{ap}_f(\tau_i(a))\cdot f_p(a)\right)^{-1}}(\mu_2(i,j,g,a))^{-1}\\
&\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1}\cdot\big(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\big)\cdot\big(\mathsf{ap}_f(\tau_i(a))\cdot f_p(a)\big)^{-1}\\
&\quad\Big\Vert\ \mathsf{ap}_{\cdots\cdot_}(\mathsf{ap}_{\left(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\right)\cdot_^{-1}}(\mathsf{ap}_{_\cdot f_p(a)}(\mathsf{ap}_{\mathsf{ap}_f}(E_3(i,j,g,a)))))^{-1}\\
&\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1}\cdot\big(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\big)\cdot\big(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)))^{-1}\cdot\tau_j(a)\cdot\mathsf{refl}_{\mathsf{inl}(a)})\cdot f_p(a)\big)^{-1}
\end{aligned}$$

$$\begin{aligned}
&\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1}\cdot\big(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\big)\cdot\big(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)))^{-1}\cdot\tau_j(a)\cdot\mathsf{refl}_{\mathsf{inl}(a)})\cdot f_p(a)\big)^{-1}\\
&\quad\Big\Vert\ \mathsf{ap}_{\cdots\cdot_}(\mathsf{ap}_{\left(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\right)\cdot_^{-1}}(\mathsf{ap}_{_\cdot f_p(a)}(\mathsf{ap}_{\mathsf{ap}_f}(E_2(i,j,g,a)))))^{-1}\\
&\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1}\cdot\big(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\big)\cdot\Big(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}\cdot\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1}\cdot\tau_j(a))\cdot f_p(a)\Big)^{-1}\\
&\quad\Big\Vert\ \mathsf{ap}_{\cdots\cdot_}(\mathsf{ap}_{\left(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\right)\cdot_^{-1}}(\mathsf{ap}_{_\cdot f_p(a)}(\mathsf{ap}_{\mathsf{ap}_f}(E_1(i,j,g,a)))))^{-1}\\
&\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1}\cdot\big(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\big)\cdot\Big(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1}\cdot\mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))\cdot\tau_j(a))\cdot f_p(a)\Big)^{-1}\\
&\quad\Big\Vert\ \mathsf{ap}_{\cdots\cdot_}(\mathsf{ap}_{\left(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\right)\cdot_^{-1}}(\Theta_{\lambda x.\,\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(x))}(f^*,a)))^{-1}\\
&\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1}\cdot\big(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\big)\cdot\Big(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1}\cdot\mathsf{ap}_{f\circ\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))\cdot\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\Big)^{-1}
\end{aligned}$$

$$\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1} \cdot \left(\mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\right) \cdot \left(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \mathsf{ap}_{f \circ \mathsf{inr} \circ \iota_j}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_f(\tau_j(a)) \cdot f_p(a)\right)^{-1}$$

$$\Big\Vert\ \mathsf{ap}_{\ldots \cdot \_}(\mu_1(i,j,g,a))^{-1}$$

$$\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1} \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))$$

$$\Big\Vert\ \mathsf{ap}_{\ldots \cdot \_}(\mathsf{ap}_{\mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}}(\rho_\psi(i,j,g,a)))^{-1}$$

$$\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1} \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)))$$

$$\Big\Vert\ \mathsf{PI}(\kappa_{i,j,g}(a))$$

$$\mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x))) = \tilde{h}(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a), \mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))})$$

$$\Big\Vert\ \mathsf{PI}(\kappa_{i,j,g}(a))$$

$$\mathsf{ap}_\psi(\kappa_{i,j,g}(a))_*(\gamma(\psi(\iota_j(a))))$$

$$\Big\Vert\ \mathsf{ap}_{-_*(\mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))})}(\rho_\psi(i,j,g,a))$$

$$\left(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))\right)_* (\mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))})$$

$$\Big\Vert\ \mathsf{PI}(\mathsf{pr}_2(F_{i,j,g})(a))$$

$$(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))_* (\gamma(\iota_j(F_{i,j,g}(\mathsf{pr}_2(F_i)(a)))))$$

$$\Big\Vert\ \mathsf{PI}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))$$

$$\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \mathsf{ap}_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))$$

$$\Big\Vert\ \mathsf{ap}_{\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \_}(\rho_{\mathsf{rec}_{\mathsf{colim}}(\mathcal{F}(E_1,E_2))}(i,j,g,\mathsf{pr}_2(F_i)(a)))$$

$$\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))$$

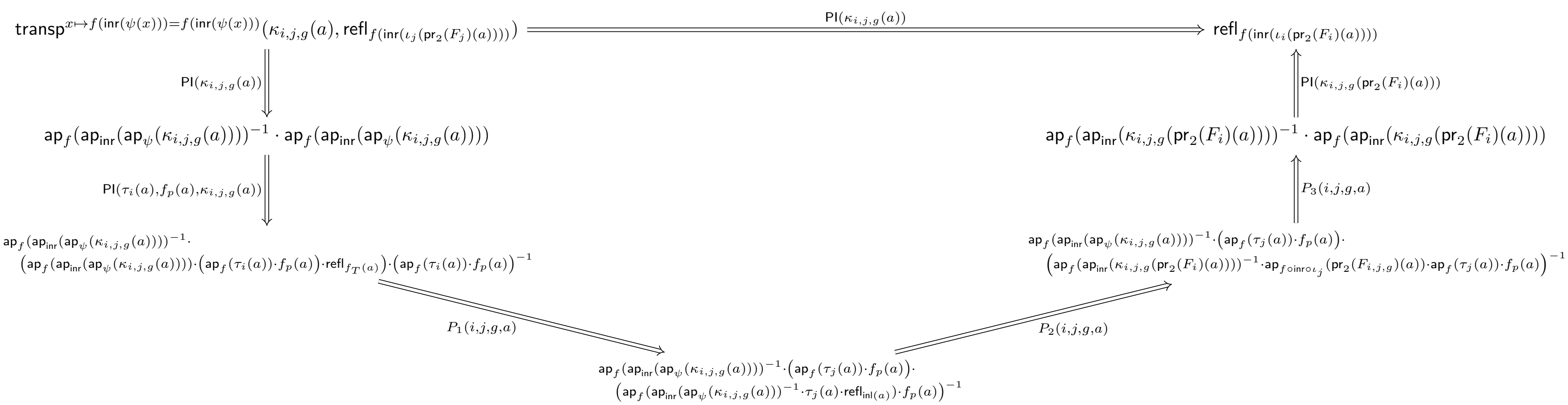
We denote these chains by $P_1(i,j,g,a)$, $P_2(i,j,g,a)$, and $P_3(i,j,g,a)$, respectively. We can show that

$$\begin{aligned} P_1(i,j,g,a) &= \mathsf{PI}(\kappa_{i,j,g}(a), \tau_j(a), f_p(a)) \\ P_3(i,j,g,a) &= \mathsf{PI}(\mathsf{pr}_2(F_{i,j,g})(a), \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)), \tau_j(a), \rho_\psi(i,j,g,a)) \end{aligned}$$

and that the square shown on the next page commutes.

$$
\begin{array}{ccc}
\begin{array}{l}
\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1}\cdot\big(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\big)\cdot \\
\quad\big(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)))^{-1}\cdot\tau_j(a)\cdot\mathsf{refl}_{\mathsf{inl}(a)})\cdot f_p(a)\big)^{-1}
\end{array}
& \overset{\mathsf{PI}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a)),\tau_j(a),f_p(a))}{=\!=\!=\!=\!=\!=} & \mathsf{refl}_{f(\mathsf{inr}(\iota_i(\mathsf{pr}_2(F_i)(a))))} \\
\Big\Downarrow {\scriptstyle P_2(i,j,g,a)} & & \Big\Uparrow {\scriptstyle \mathsf{PI}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))} \\
\begin{array}{l}
\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))))^{-1}\cdot\big(\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\big)\cdot \\
\quad\big(\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1}\cdot\mathsf{ap}_{f\circ\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))\cdot\mathsf{ap}_f(\tau_j(a))\cdot f_p(a)\big)^{-1}
\end{array}
& \underset{\mathsf{PI}(\mathsf{pr}_2(F_{i,j,g})(a),\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)),\tau_j(a),\rho_\psi(i,j,g,a))}{\Longrightarrow} & \mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))^{-1}\cdot\mathsf{ap}_f(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))))
\end{array}
$$

These three equalities together give us a commuting diagram



It's easy to check that the bottom string of paths in this pentagon equals $\chi(\mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))})^{-1} \cdot \mathsf{apd}_{\gamma(\psi(-))}(\kappa_{i,j,g}(a))$, so that

$$\chi(\mathsf{refl}_{f(\mathsf{inr}(\iota_j(\mathsf{pr}_2(F_j)(a))))})^{-1} \cdot \mathsf{apd}_{\gamma(\psi(-))}(\kappa_{i,j,g}(a)) \ = \ \mathsf{Pl}(\kappa_{i,j,g}(a))$$

It follows that

$$\begin{array}{c}
\mathsf{apd}_{\left(\mathsf{ap}_f(\mathsf{glue}_{\mathcal{P}_A(F)}(-))^{-1} \cdot f_p([\mathsf{id}_A](-))\right)\cdot\sigma(-)}(\kappa_{i,j,g}(a))^{-1} \cdot \\
\mathsf{ap}_{\mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x)))=\tilde{h}(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a),-)}(\mathsf{Pl}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)), f_p(a))) \cdot \mathsf{apd}_{\gamma(\psi(-))}(\kappa_{i,j,g}(a)) \\
\Vert \\
\mathsf{Pl}(\kappa_{i,j,g}(a)) \cdot \mathsf{ap}_{\mathsf{transp}^{x \mapsto f(\mathsf{inr}(\psi(x)))=f(\mathsf{inr}(\psi(x)))}(\kappa_{i,j,g}(a),-)}(\mathsf{Pl}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_j(a)), f_p(a))) \cdot \mathsf{Pl}(\kappa_{i,j,g}(a)) \\
\Vert \, {\scriptstyle \mathsf{Pl}(\kappa_{i,j,g}(a))} \\
\mathsf{Pl}(\mathsf{glue}_{\mathcal{P}_A(F)}(\iota_i(a)), f_p(a))
\end{array}$$

as desired.

∎

Corollary 5.4.4. *Pointed acyclic types are closed under* colim^*_Γ.[4]

Proof. Since $\Sigma : \mathcal{U}^* \to \mathcal{U}^*$ is a 2-coherent left adjoint (Example B.0.3), we have that $\Sigma(\mathsf{colim}^*_\Gamma(F)) \simeq \mathsf{colim}^*_{i:\Gamma}(\Sigma(F_i))$. If each F_i is acyclic, then the second colimit is the colimit of the constant pointed diagram at $\mathbf{1}$, which is trivial as the cofiber of the identity function on $\mathsf{colim}_\Gamma \mathbf{1}$. ∎

Lemma 5.4.5. *For every map* $h^* : T \to_A U$*, the square*

$$\begin{array}{ccc}
(\mathsf{colim}\, F \to_A T) & \xrightarrow{h^* \circ -} & (\mathsf{colim}\, F \to_A U) \\
{\scriptstyle e_{F,T}} \downarrow & & \downarrow {\scriptstyle e_{F,U}} \\
\mathsf{Cocone}_F(T) & \xrightarrow[\mathsf{Cocone}_F(f^* \circ -)]{} & \mathsf{Cocone}_F(U)
\end{array}$$

commutes where $\mathsf{Cocone}_F(f^* \circ -)$ *is defined by*

$$(x, R) \mapsto \Big(\lambda i. h^* \circ x_i, \lambda i \lambda j \lambda g. \Big(\lambda x. \mathsf{ap}_h(\mathsf{pr}_1(R_{i,j,g})(x)), \lambda a. \Theta_{\mathsf{pr}_1(R_{i,j,g})}(h^*, a) \cdot \mathsf{ap}_{\mathsf{ap}_h(-)\cdot h_p(a)}(\mathsf{pr}_2(R_{i,j,g})(a))\Big)\Big).$$ [5]

[4] Recall that a type is *acyclic* if its suspension is contractible (see [3]).
[5] This is formalized in [7, `Colimit-code/Map-Nat/CosColimitPstCmp.agda`].

Action on maps

We now describe the action of $\mathsf{colim}^A_\Gamma(-)$ on morphisms. Suppose that F and G are A-diagrams over Γ. Consider a morphism $\delta :\equiv \big(d, \langle \xi, \tilde{\xi} \rangle\big) : F \Rightarrow_A G$

$$\begin{array}{ccc} F_i & \xrightarrow{F_{i,j,g}} & F_j \\ {\scriptstyle d_i}\big\downarrow & \langle \xi_{i,j,g}, \tilde{\xi}_{i,j,g} \rangle & \big\downarrow{\scriptstyle d_j} \\ G_i & \xrightarrow[G_{i,j,g}]{} & G_j \end{array}$$

of A-diagrams. We have a commuting square

$$\begin{array}{ccc} F_i & \xrightarrow{\quad d_i \quad} & G_i \\ {\scriptstyle \iota^F_i}\big\downarrow & & \big\downarrow{\scriptstyle \iota^G_i} \\ \mathsf{colim}^A_\Gamma(F) & \xrightarrow[\mathsf{colim}^A_\Gamma(\delta)]{} & \mathsf{colim}^A_\Gamma(G) \end{array}$$

Indeed, we have a function $\mathsf{colim}_\Gamma(\mathcal{F}(F)) \xrightarrow{\hat{\delta}} \mathsf{colim}_\Gamma(\mathcal{F}(G))$ induced by the map

$$\begin{array}{ccc} \mathsf{pr}_1(F_i) & \xrightarrow{\mathsf{pr}_1(F_{i,j,g})} & \mathsf{pr}_1(F_j) \\ {\scriptstyle \mathsf{pr}_1(d_i)}\big\downarrow & \xi_{i,j,g} & \big\downarrow{\scriptstyle \mathsf{pr}_1(d_j)} \\ \mathsf{pr}_1(G_i) & \xrightarrow[\mathsf{pr}_1(G_{i,j,g})]{} & \mathsf{pr}_1(G_j) \end{array}$$

of diagrams over Γ. Note that for each $a : A$,

$$\tilde{\xi}_{i,j,g}(a) \;:\; \xi_{i,j,g}(\mathsf{pr}_2(F_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(G_{i,j,g})}(\mathsf{pr}_2(d_i)(a)) \cdot \mathsf{pr}_2(G_{i,j,g})(a) = \mathsf{ap}_{\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{pr}_2(d_j)(a)$$

Without loss of generality, we may assume that $\tilde{\xi}_{i,j,g}(a)$ instead has type

$$\xi_{i,j,g}(\mathsf{pr}_2(F_i)(a)) = \underbrace{\mathsf{ap}_{\mathsf{pr}_1(G_{i,j,g})}(\mathsf{pr}_2(d_i)(a)) \cdot \mathsf{pr}_2(G_{i,j,g})(a) \cdot \mathsf{pr}_2(d_j)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1}}_{E_{i,j,g}(a)}$$

Now, the triangle

$$\begin{array}{ccc} & \mathsf{colim}_\Gamma A & \\ {\scriptstyle \psi_F}\swarrow & & \searrow{\scriptstyle \psi_G} \\ \mathsf{colim}_\Gamma(\mathcal{F}(F)) & \xrightarrow[\hat{\delta}]{} & \mathsf{colim}_\Gamma(\mathcal{F}(G)) \end{array}$$

commutes by induction on $\mathsf{colim}_\Gamma A$. Indeed, we have a path

$$\hat{\delta}(\psi_F(\iota_i(a))) = \hat{\delta}(\iota_i(\mathsf{pr}_2(F_i)(a))) = \iota_i(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a))) \overset{C_i(a) :\equiv \mathsf{ap}_{\iota_i}(\mathsf{pr}_2(d_i)(a))}{=\!=\!=} \iota_i(\mathsf{pr}_2(G_i)(a)) = \psi_G(\iota_i(a))$$

for all $i : \Gamma_0$ and $a : A$. By Lemma 5.4.1, we have a path

$$S_{i,j,g}(a) \;:\; \kappa_{i,j,g}(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_{\iota_j \circ \mathsf{pr}_1(G_{i,j,g})}(\mathsf{pr}_2(d_i)(a)) \cdot \kappa_{i,j,g}(\mathsf{pr}_2(G_i)(a)) = C_i(a)$$

Hence we have a chain $\gamma_{i,j,g}(a)$ of equalities

$$\begin{array}{c}
(\kappa_{i,j,g}(a))_*(C_j(a)) \\
\Vert \\
{\scriptstyle \mathsf{Pl}(\kappa_{i,j,g}(a), C_j(a))} \\
\Vert \\
\mathsf{ap}_{\hat{\delta}}(\mathsf{ap}_{\psi_F}(\kappa_{i,j,g}(a)))^{-1} \cdot C_j(a) \cdot \mathsf{ap}_{\psi_G}(\kappa_{i,j,g}(a)) \\
\Vert \\
{\scriptstyle \textit{via } \rho_{\psi_F}(i,j,g,a)} \\
\Vert \\
\mathsf{ap}_{\hat{\delta}}(\mathsf{ap}_{\iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot C_j(a) \cdot \mathsf{ap}_{\psi_G}(\kappa_{i,j,g}(a)) \\
\Vert \\
{\scriptstyle \mathsf{Pl}(\mathsf{pr}_2(F_{i,j,g}(a)))} \\
\Vert \\
\mathsf{ap}_{\hat{\delta}}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_{\iota_j \circ \mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot C_j(a) \cdot \mathsf{ap}_{\psi_G}(\kappa_{i,j,g}(a)) \\
\Vert \\
{\scriptstyle \textit{via } \rho_{\hat{\delta}}(i,j,g,\mathsf{pr}_2(F_i)(a))} \\
\Vert \\
\Big(\mathsf{ap}_{\iota_j}(\xi_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a)))\Big)^{-1} \cdot \mathsf{ap}_{\iota_j \circ \mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot C_j(a) \cdot \mathsf{ap}_{\psi_G}(\kappa_{i,j,g}(a)) \\
\Vert \\
{\scriptstyle \textit{via } \tilde{\xi}_{i,j,g}(a)} \\
\Vert \\
\Big(\mathsf{ap}_{\iota_j}(E_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a)))\Big)^{-1} \cdot \mathsf{ap}_{\iota_j \circ \mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot C_j(a) \cdot \mathsf{ap}_{\psi_G}(\kappa_{i,j,g}(a)) \\
\Vert \\
{\scriptstyle \textit{via } \rho_{\psi_G}(i,j,g,a)} \\
\Vert \\
\Big(\mathsf{ap}_{\iota_j}(E_{i,j,g}(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a)))\Big)^{-1} \cdot \mathsf{ap}_{\iota_j \circ \mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot C_j(a) \cdot \mathsf{ap}_{\iota_j}(\mathsf{pr}_2(G_{i,j,g})(a))^{-1} \cdot \kappa_{i,j,g}(\mathsf{pr}_2(G_i)(a)) \\
\Vert \\
{\scriptstyle \mathsf{Pl}(\mathsf{pr}_2(d_i)(a), \mathsf{pr}_2(G_{i,j,g})(a), \mathsf{pr}_2(d_j)(a), \mathsf{pr}_2(F_{i,j,g})(a), \kappa_{i,j,g}(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a))))} \\
\Vert \\
\kappa_{i,j,g}(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_{\iota_j \circ \mathsf{pr}_1(G_{i,j,g})}(\mathsf{pr}_2(d_i)(a)) \cdot \kappa_{i,j,g}(\mathsf{pr}_2(G_i)(a)) \\
\Vert \\
{\scriptstyle S_{i,j,g}(a)} \\
\Vert \\
C_i(a)
\end{array}$$

for all $i, j : \Gamma_0$, $g : \Gamma_1(i,j)$, and $a : A$. This proves that the triangle commutes.

We now have a map of spans

$$\begin{array}{ccccc}
A & \longleftarrow & \mathsf{colim}_\Gamma\, A & \longrightarrow & \mathsf{colim}_\Gamma(\mathcal{F}(F)) \\
\Big\downarrow {\scriptstyle \mathsf{id}} & \mathsf{refl}_{[\mathsf{id}](x)} & \Big\downarrow {\scriptstyle \mathsf{id}} & C^{-1} & \Big\downarrow {\scriptstyle \bar{\delta}} \\
A & \longleftarrow & \mathsf{colim}_\Gamma\, A & \longrightarrow & \mathsf{colim}_\Gamma(\mathcal{F}(G))
\end{array}$$

This gives us the data

$$\begin{aligned}
\mathsf{colim}^A_\Gamma(\delta) &:= \big(\Psi_\delta, \lambda a.\mathsf{refl}_{\mathsf{inl}(a)}\big) : \mathcal{P}_A(F) \to_A \mathcal{P}_A(G) \\
\Psi_\delta(\mathsf{inr}(\iota_i(x))) &\equiv \mathsf{inr}(\iota_i(\mathsf{pr}_1(d_i)(x))) \\
\rho_{\Psi_\delta}(x) &: \mathsf{ap}_{\Psi_\delta}(\mathsf{glue}_{\mathcal{P}_A(F)}(x)) = \mathsf{glue}_{\mathcal{P}_A(G)}(x) \cdot \mathsf{ap}_{\mathsf{inr}}(C^{-1}(x))
\end{aligned}$$

Corollary 5.4.6. *The forgetful functor $A/\mathcal{U} \to \mathcal{U}$ creates colimits over trees.*

Proof. Suppose that Γ is a tree and let F be an A-diagram over Γ. The function $[\mathsf{id}_A] : \mathsf{colim}_\Gamma A \to A$ is an equivalence. One can check that

$$\begin{array}{ccc}
\mathsf{colim}_\Gamma A & \xrightarrow{\psi} & \mathsf{colim}_\Gamma(\mathcal{F}(F)) \\
{\scriptstyle [\mathsf{id}_A]}\downarrow & & \downarrow{\scriptstyle \mathsf{id}} \\
A & \xrightarrow[\psi \circ [\mathsf{id}_A]^{-1}]{} & \mathsf{colim}_\Gamma(\mathcal{F}(F))
\end{array}$$

is a pushout square. This gives us an equivalence $\gamma : \mathcal{P}_A(F) \xrightarrow{\simeq} \mathsf{colim}_\Gamma(\mathcal{F}(F))$ such that

$$\gamma(\mathsf{inr}(\iota_i(x))) \equiv \iota_i(x).$$

for all $i : \Gamma_0$ and $x : \mathsf{pr}_1(F_i)$. We also see that

$$\mathsf{ap}_\gamma(\mathsf{ap}_{\mathsf{inr}}(\kappa_{i,j,g}(x))) = \mathsf{ap}_{\gamma \circ \mathsf{inr}}(\kappa_{i,j,g}(x)) \equiv \mathsf{ap}_{\mathsf{id}}(\kappa_{i,j,g}(x)) = \kappa_{i,j,g}(x)$$

for all $i, j : \Gamma_0$, $g : \Gamma_1(i,j)$, and $x : \mathsf{pr}_1(F_i)$. This means that γ is a morphism of cocones under $\mathcal{F}(F)$. It follows that the forgetful functor preserves colimits over Γ.

It remains to prove that the forgetful functor reflects colimits over Γ. Consider a cocone $\mathcal{C}$

$$\begin{array}{ccc}
F_i & \xrightarrow{F_{i,j,g}} & F_j \\
{\scriptstyle r_i}\searrow & \langle H, K\rangle & \swarrow{\scriptstyle r_j} \\
& C &
\end{array}$$

under F as well as the cocone $\mathcal{F}(\mathcal{C}) := (\mathsf{pr}_1(C), \mathsf{pr}_1 \circ r, H)$ under $\mathcal{F}(F)$ obtained by applying the forgetful functor to $\mathcal{C}$. Suppose that $\mathcal{F}(\mathcal{C})$ is colimiting in $\mathcal{U}$. By the universal property of colimits in $A/\mathcal{U}$, we have a morphism $(\mathcal{P}_A(F), \mathsf{inl}) \xrightarrow{\tau} C$ of cocones, which induces a morphism $\mathsf{pr}_1(\mathcal{P}_A(F)) \xrightarrow{\mathcal{F}(\tau)} \mathsf{pr}_1(C)$ of cocones in $\mathcal{U}$. This morphism is unique by the universal property of colimits. Moreover, by Proposition 5.2.3, there exists a cocone equivalence from $\mathsf{pr}_1(\mathcal{P}_A(F))$ to $\mathsf{pr}_1(C)$ as both of them are colimiting. It follows that $\mathcal{F}(\tau)$ must be an equivalence. Thus, τ is a cocone morphism whose underlying function $\mathcal{P}_A(F) \to \mathsf{pr}_1(C)$ of types is an equivalence. This means that τ is a cocone equivalence, so that $\mathcal{C}$ is colimiting. $\square$

Question 5.4.7. *Let Δ be a graph and G be an A-diagram over Δ. If the canonical function $\mathsf{colim}_\Delta(\mathcal{F}(G)) \to \mathsf{pr}_1(\mathsf{colim}^A_\Delta(G))$ is an equivalence, then is Δ a tree?*

Corollary 5.4.8. *If Γ is a tree, then for each $X : A/\mathcal{U}$, the colimit colim^A_Γ of the constant diagram at X is X.*

Proof. This is an easy consequence of Corollary 5.1.3. □

Note 5.4.9. Thanks to Lemma 3.3.7, we can refine Corollary 5.4.6 as follows. If $|\Gamma|$ is n-connected, then so is the underlying function of the cocone morphism $\mathsf{colim}_\Gamma(\mathcal{F}(F)) \xrightarrow{\mathsf{inr}} \mathcal{P}_A(F)$ in $\mathcal{U}$. Thus, the degree to which $\mathcal{F}$ approximates $\mathsf{colim}^A_\Gamma(F)$ increases linearly with how close Γ is to a tree.

Adjunction

Next, we verify that our functor colim^A_Γ is correct by showing that it's left adjoint (in the sense of Definition 3.1.7) to the constant diagram functor. Consider again a morphism $\delta :\equiv \left(d, \langle \xi, \tilde{\xi} \rangle\right) : F \Rightarrow_A G$ of A-diagrams.

Note 5.4.10. Consider the A-cocone $K(\delta)$

$$\left(\lambda i.\, \left(\mathsf{inr} \circ \iota_i \circ \mathsf{pr}_1(d_i), \lambda a.\, \mathsf{ap}_{\mathsf{inr}\circ\iota_i}(\mathsf{pr}_2(d_i)(a)) \cdot \tau^G_i(a)\right), \lambda i \lambda j \lambda g.\, \left(\lambda x.\, \mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\xi_{i,j,g}(x))^{-1} \cdot \kappa^G_{i,j,g}(\mathsf{pr}_1(d_i)(x))), \lambda a.\Theta(\epsilon_{i,j,g}(a), \tilde{\xi}_{i,j,g}(a))\right)\right)$$

on $\mathcal{P}_A(G)$ under F where $\Theta(\epsilon_{i,j,g}(a), \tilde{\xi}_{i,j,g}(a))$ denotes the chain of equalities

$$\begin{gathered}
\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(\xi_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \kappa^G_{i,j,g}(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j\circ\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(d_j)(a)) \cdot \tau^G_j(a) \\
\Big\| \;\textit{via } \tilde{\xi}_{i,j,g}(a) \\
\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(E_{i,j,g}(a))^{-1} \cdot \kappa^G_{i,j,g}(\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a))))^{-1} \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j\circ\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(d_j)(a)) \cdot \tau^G_j(a) \\
\Big\| \;\textit{via Lemma 5.4.1 applied to } \kappa^G_{i,j,g} \textit{ and } \mathsf{pr}_2(d_i)(a) \\
\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{\iota_j}(E_{i,j,g}(a))^{-1} \cdot \mathsf{ap}_{\iota_j\circ\mathsf{pr}_1(G_{i,j,g})}(\mathsf{pr}_2(d_i)(a)) \cdot \kappa^G_{i,j,g}(\mathsf{pr}_2(G_i)(a)) \cdot \mathsf{ap}_{\iota_i}(\mathsf{pr}_2(d_i)(a))^{-1})^{-1} \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j\circ\mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(d_j)(a)) \cdot \tau^G_j(a) \\
\Big\| \;\mathsf{PI}(\mathsf{pr}_2(F_{i,j,g})(a), \mathsf{pr}_2(d_j)(a), \mathsf{pr}_2(d_i)(a), \kappa^G_{i,j,g}(\mathsf{pr}_2(G_i)(a)), \mathsf{pr}_2(G_{i,j,g})(a)) \\
\mathsf{ap}_{\mathsf{inr}\circ\iota_i}(\mathsf{pr}_2(d_i)(a)) \cdot \mathsf{ap}_{\mathsf{inr}}(\kappa^G_{i,j,g}(\mathsf{pr}_2(G_i)(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{inr}\circ\iota_j}(\mathsf{pr}_2(G_{i,j,g})(a)) \cdot \tau^G_j(a) \\
\Big\| \;\mathsf{ap}_{\mathsf{ap}_{\mathsf{inr}\circ\iota_i}(\mathsf{pr}_2(d_i)(a)) \cdot -}(\epsilon_{i,j,g}(a)) \\
\mathsf{ap}_{\mathsf{inr}\circ\iota_i}(\mathsf{pr}_2(d_i)(a)) \cdot \tau^G_i(a)
\end{gathered}$$

We claim that

$$\mathsf{colim}_\Gamma(\delta) = e^{-1}_{F,\mathsf{colim}(G)}(K(\delta)) \qquad (\texttt{map-eq})$$

It suffices to show that $\mathsf{colim}_\Gamma(\delta)$ belongs to the fiber of $e_{F,\mathsf{colim}(G)}$ over $K(\delta)$. The proof closely resembles the first half of the proof of Theorem 5.4.3. We again leave it to the Agda formalization (see [7, `Colimit-code/Map-Nat/CosColimitMap16.agda`]).

Definition 5.4.11. For each $T : \mathsf{Ob}(A/\mathcal{U})$, define $\mathsf{Cocone}^T(- \circ \delta) : \mathsf{Cocone}_G(T) \to \mathsf{Cocone}_F(T)$ by

$$(x, \langle R_1, R_2 \rangle)$$
$$\downarrow$$
$$\Big(\lambda i.\, \Big(\mathsf{pr}_1(x_i) \circ \mathsf{pr}_1(d_i), \lambda a.\, \mathsf{ap}_{\mathsf{pr}_1(x_i)}(\mathsf{pr}_2(d_i)(a)) \cdot \mathsf{pr}_2(x_i)(a)\Big), \lambda i \lambda j \lambda g.\, \Big(\lambda x.\, \mathsf{ap}_{\mathsf{pr}_1(x_j)}(\xi_{i,j,g}(x))^{-1} \cdot R_1(i,j,g,\mathsf{pr}_1(d_i)(x)), \lambda a. V_{x,R_1,R_2}(i,j,g,a)\Big)\Big)$$

where $V_{x,R_1,R_2}(i,j,g,a)$ denotes the chain of equalities

$$\Big(\mathsf{ap}_{\mathsf{pr}_1(x_j)}(\xi_{i,j,g}(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot R_1(i,j,g,\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a)))\Big)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j) \circ \mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)}(\mathsf{pr}_2(d_j)(a)) \cdot \mathsf{pr}_2(x_j)(a)$$
$$\Vert$$
$$\mathsf{PI}(R_1(i,j,g,\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a))), \mathsf{ap}_{\mathsf{pr}_1(x_j)}(\xi_{i,j,g}(\mathsf{pr}_2(F_i)(a))))$$
$$\Vert$$
$$R_1(i,j,g,\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)}(\xi_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j) \circ \mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)}(\mathsf{pr}_2(d_j)(a)) \cdot \mathsf{pr}_2(x_j)(a)$$
$$\Vert$$
$$\mathsf{ap}_{R_1(i,j,g,\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)}(-) \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j) \circ \mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)}(\mathsf{pr}_2(d_j)(a)) \cdot \mathsf{pr}_2(x_j)(a)}(\tilde{\xi}_{i,j,g}(a))$$
$$\Vert$$
$$R_1(i,j,g,\mathsf{pr}_1(d_i)(\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)}(E_{i,j,g}(a)) \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j) \circ \mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)}(\mathsf{pr}_2(d_j)(a)) \cdot \mathsf{pr}_2(x_j)(a)$$
$$\Vert$$
$$\textit{via Lemma 5.4.1 applied to } R_1 \textit{ and } \mathsf{pr}_2(d_i)(a)$$
$$\Vert$$
$$\Big(\mathsf{ap}_{\mathsf{pr}_1(x_i)}(\mathsf{pr}_2(d_i)(a)) \cdot R_1(i,j,g,\mathsf{pr}_2(G_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)}(\mathsf{ap}_{\mathsf{pr}_1(G_{i,j,g})}(\mathsf{pr}_2(d_i)(a)))^{-1}\Big) \cdot$$
$$\mathsf{ap}_{\mathsf{pr}_1(x_j)}(E_{i,j,g}(a)) \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j) \circ \mathsf{pr}_1(d_j)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)}(\mathsf{pr}_2(d_j)(a)) \cdot \mathsf{pr}_2(x_j)(a)$$
$$\Vert$$
$$\mathsf{PI}(\mathsf{ap}_{\mathsf{pr}_1(x_i)}(\mathsf{pr}_2(d_i)(a)), \mathsf{ap}_{\mathsf{pr}_1(G_{i,j,g})}(\mathsf{pr}_2(d_i)(a)), \mathsf{pr}_2(G_{i,j,g})(a), \mathsf{pr}_2(d_j)(a), \mathsf{pr}_2(F_{i,j,g})(a), R_1(i,j,g,\mathsf{pr}_2(G_i)(a))^{-1})$$
$$\Vert$$
$$\mathsf{ap}_{\mathsf{pr}_1(x_i)}(\mathsf{pr}_2(d_i)(a)) \cdot R_1(i,j,g,\mathsf{pr}_2(G_i)(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(x_j)}(\mathsf{pr}_2(G_{i,j,g})(a)) \cdot \mathsf{pr}_2(x_j)(a)$$
$$\Vert$$
$$\mathsf{ap}_{\mathsf{ap}_{\mathsf{pr}_1(x_i)}(\mathsf{pr}_2(d_i)(a)) \cdot -}(R_2(i,j,g,a))$$
$$\Vert$$
$$\mathsf{ap}_{\mathsf{pr}_1(x_i)}(\mathsf{pr}_2(d_i)(a)) \cdot \mathsf{pr}_2(x_i)(a)$$

Lemma 5.4.12. *The square*

$$\begin{array}{ccc}
(\mathsf{colim}(G) \to_A T) & \xrightarrow{- \circ \mathsf{colim}^A_\Gamma(\delta)} & (\mathsf{colim}(F) \to_A T) \\
{\scriptstyle e_{G,T}} \downarrow & & \downarrow {\scriptstyle e_{F,T}} \\
\mathsf{Cocone}_G(T) & \xrightarrow[\mathsf{Cocone}^T(- \circ \delta)]{} & \mathsf{Cocone}_F(T)
\end{array}$$

commutes.[6]

[6]This is formalized in [7, `Colimit-code/Map-Nat/CosColimitPreCmp.agda`].

Proof. For each $f^* : \mathsf{colim}(G) \to_A T$, note that

$$\begin{aligned}
& e_{F,T}(f^* \circ e^{-1}_{F,\mathsf{colim}(G)}(K(\delta))) \\
&= \mathsf{Cocone}_F(f^* \circ -)(e_{F,\mathsf{colim}(G)}(e^{-1}_{F,\mathsf{colim}(G)}(K(\delta)))) && \text{(Lemma 5.4.5)} \\
&= \mathsf{Cocone}_F(f^* \circ -)(K(\delta)).
\end{aligned}$$

Thanks to (`map-eq`), it thus suffices to prove that

$$\mathsf{Cocone}_F(f^* \circ -)(K(\delta)) \;=\; \mathsf{Cocone}^T(- \circ \delta)(e_{G,T}(f^*))$$

We leave such a proof, which is messy yet routine, to the Agda formalization (see [7, `Colimit-code/Map-Nat/CosColimitMap18.agda`]). ☐

Corollary 5.4.13. *We have an adjunction* $\mathsf{colim}^A_\Gamma \dashv \mathsf{const}_\Gamma$*, where* const_Γ *denotes the constant diagram functor* $A/\mathcal{U} \to \mathsf{Diag}_A(\Gamma)$*.*[7]

5.5 Second construction of coslice colimits

In this section, we apply the 3×3 lemma to our first construction of $\mathsf{colim}^A_\Gamma(F)$ to obtain the familiar construction of $\mathsf{colim}^A_\Gamma(F)$ as a pushout of coproducts in $A/\mathcal{U}$.

To begin, consider the following grid of commuting squares:

$$\begin{array}{ccccc}
\sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)} \mathsf{pr}_1(F_i) & \xleftarrow{\mathsf{id}+\mathsf{id}} & \Big(\sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)} \mathsf{pr}_1(F_i)\Big) + \Big(\sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)} \mathsf{pr}_1(F_i)\Big) & \xrightarrow{(i,x)+(j,\mathsf{pr}_1(F_{i,j,g})(x))} & \sum_{i:\Gamma_0} \mathsf{pr}_1(F_i) \\
\uparrow{\scriptstyle (i,j,g,\mathsf{pr}_2(F_i)(a))} & \mathsf{refl}_{(i,j,g,\mathsf{pr}_2(F_i)(a))}+\mathsf{refl}_{(i,j,g,\mathsf{pr}_2(F_i)(a))} & \uparrow{\scriptstyle (i,j,g,\mathsf{pr}_2(F_i)(a))+(i,j,g,\mathsf{pr}_2(F_i)(a))} & \mathsf{refl}_{(i,\mathsf{pr}_2(F_i)(a))}+\mathsf{ap}_{(j,-)}(\mathsf{pr}_2(F_{i,j,g})(a)) & \uparrow{\scriptstyle (i,\mathsf{pr}_2(F_i)(a))} \\
\Big(\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)\Big) \times A & \xleftarrow{\mathsf{id}+\mathsf{id}} & \Big(\Big(\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)\Big) \times A\Big) + \Big(\Big(\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)\Big) \times A\Big) & \xrightarrow{(i,a)+(j,a)} & \Gamma_0 \times A \\
\downarrow{\scriptstyle \mathsf{pr}_2} & \mathsf{refl}_a+\mathsf{refl}_a & \downarrow{\scriptstyle \mathsf{pr}_2+\mathsf{pr}_2} & \mathsf{refl}_a+\mathsf{refl}_a & \downarrow{\scriptstyle \mathsf{pr}_2} \\
A & \xleftarrow[\mathsf{id}_A]{} & A & \xrightarrow[\mathsf{id}_A]{} & A
\end{array}$$

Call the pushouts of the left, middle, and right vertical spans V_1, V_2, and V_3, respectively. Call the pushouts of the top, middle, and bottom horizontal spans H_1, H_2, and H_3, respectively. We can form two additional pushouts

$$\begin{array}{ccc}
V_2 & \xrightarrow{\delta_2} & V_3 \\
{\scriptstyle \delta_1}\downarrow & \ulcorner & \downarrow \\
V_1 & \longrightarrow & P_V
\end{array}
\qquad\qquad
\begin{array}{ccc}
H_2 & \xrightarrow{\eta_1} & H_1 \\
{\scriptstyle \eta_2}\downarrow & \ulcorner & \downarrow \\
H_3 & \longrightarrow & P_H
\end{array}$$

where

[7] This is formalized in [7, `Colimit-code/Main-Theorem/CosColim-Adjunction.agda`].

- δ_1 denotes the function induced by the middle-to-left map of spans;
- δ_2 the function induced by the middle-to-right map of spans;
- η_1 the function induced by the middle-to-top map of spans; and
- η_2 the function induced by the middle-to-bottom map of spans.

Licata and Brunerie construct an equivalence $\tau_1 : P_H \xrightarrow{\simeq} P_V$ of types by double induction on pushouts [8, Section VII], which satisfies, in particular,

$$\begin{aligned} \tau_1(\mathsf{inl}(\mathsf{inl}(a))) &\equiv \mathsf{inl}(\mathsf{inl}(a)) \\ \tau_1(\mathsf{inr}(\mathsf{inr}(i,x))) &\equiv \mathsf{inr}(\mathsf{inr}(i,x)). \end{aligned}$$

Lemma 5.5.1. *We have an equivalence*

$$\xi \ : \ V_2 \xrightarrow{\simeq} \left(\bigvee_{i,j,g} \mathsf{pr}_1(F_i)\right) \vee \left(\bigvee_{i,j,g} \mathsf{pr}_1(F_i)\right)$$

Proof. Define ξ by pushout recursion with the commuting square

$$\begin{array}{ccc} \left(\left(\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)\right)\times A\right) + \left(\left(\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)\right)\times A\right) & \longrightarrow & \left(\sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)}\mathsf{pr}_1(F_i)\right) + \left(\sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)}\mathsf{pr}_1(F_i)\right) \\ \Big\downarrow & \mathsf{ap}_{\mathsf{inl}}(\mathsf{glue}_{\bigvee_{i,j,g}\mathsf{pr}_1(F_i)})(i,j,g,a) \ + \ \mathsf{glue}_{\bigvee\vee\bigvee}(a)\cdot\mathsf{ap}_{\mathsf{inr}}(\mathsf{glue}_{\bigvee_{i,j,g}\mathsf{pr}_1(F_i)})(i,j,g,a) & \Big\downarrow{\scriptstyle \mathsf{inl}\circ\mathsf{inr}+\mathsf{inr}\circ\mathsf{inr}} \\ A & \xrightarrow[a\mapsto\mathsf{inl}(\mathsf{inl}(a))]{} & \left(\bigvee_{i,j,g}\mathsf{pr}_1(F_i)\right)\vee\left(\bigvee_{i,j,g}\mathsf{pr}_1(F_i)\right) \end{array}$$

Define a quasi-inverse $\tilde{\xi}$ of ξ by recursion on $\bigvee \vee \bigvee$ with the commuting square

$$\begin{array}{ccc} A & \longrightarrow & \bigvee_{i,j,g}\mathsf{pr}_1(F_i) \\ \Big\downarrow & \mathsf{refl}_{\mathsf{inl}(a)} & \Big\downarrow{\scriptstyle \epsilon_2} \\ \bigvee_{i,j,g}\mathsf{pr}_1(F_i) & \xrightarrow[\epsilon_1]{} & V_2 \end{array}$$

Here, ϵ_1 and ϵ_2 are induced by the commuting squares

$$\begin{array}{ccc}
\left(\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)\right)\times A & \longrightarrow & \sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)}\mathsf{pr}_1(F_i) \\
\Big\downarrow & {\scriptstyle (i,j,g,a)\mapsto\mathsf{glue}_{V_2}(\mathsf{inl}(i,j,g,a))} & \Big\downarrow{\scriptstyle t\mapsto\mathsf{inr}(\mathsf{inl}(t))} \\
A & \xrightarrow[a\mapsto\mathsf{inl}(a)]{} & V_2
\end{array}$$

$$\begin{array}{ccc}
\left(\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)\right)\times A & \longrightarrow & \sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0}\Gamma_1(i,j)}\mathsf{pr}_1(F_i) \\
\Big\downarrow & {\scriptstyle (i,j,g,a)\mapsto\mathsf{glue}_{V_2}(\mathsf{inr}(i,j,g,a))} & \Big\downarrow{\scriptstyle t\mapsto\mathsf{inr}(\mathsf{inr}(t))} \\
A & \xrightarrow[a\mapsto\mathsf{inl}(a)]{} & V_2
\end{array}$$

respectively.

By induction on V_2, we prove that $\tilde{\xi}\circ\xi\sim\mathsf{id}_{V_2}$. We have that

$$\begin{aligned}
\tilde{\xi}(\xi(\mathsf{inl}(a))) &\equiv \tilde{\xi}(\mathsf{inl}(\mathsf{inl}(a))) \\
&\equiv \epsilon_1(\mathsf{inl}(a)) \\
&\equiv \mathsf{inl}(a) \\
\tilde{\xi}(\xi(\mathsf{inr}(\mathsf{inl}(i,j,g,x)))) &\equiv \tilde{\xi}(\mathsf{inl}(\mathsf{inr}(i,j,g,x))) \\
&\equiv \epsilon_1(\mathsf{inr}(i,j,g,x)) \\
&\equiv \mathsf{inr}(\mathsf{inl}(i,j,g,x)) \\
\tilde{\xi}(\xi(\mathsf{inr}(\mathsf{inr}(i,j,g,x)))) &\equiv \tilde{\xi}(\mathsf{inr}(\mathsf{inr}(i,j,g,x))) \\
&\equiv \epsilon_2(\mathsf{inr}(i,j,g,x)) \\
&\equiv \mathsf{inr}(\mathsf{inr}(i,j,g,x)) \\
\mathsf{transp}^{x\mapsto\tilde{\xi}(\xi(x))=x}(\mathsf{glue}_{V_2}(\mathsf{inl}(i,j,g,a)),\mathsf{refl}_{\mathsf{inl}(a)}) &= \mathsf{ap}_{\tilde{\xi}}(\mathsf{ap}_{\xi}(\mathsf{glue}(\mathsf{inl}(i,j,g,a))))^{-1}\cdot\mathsf{glue}(\mathsf{inl}(i,j,g,a)) \\
&= \mathsf{glue}(\mathsf{inl}(i,j,g,a))^{-1}\cdot\mathsf{glue}(\mathsf{inl}(i,j,g,a)) \\
&= \mathsf{refl}_{\mathsf{inr}(\mathsf{inl}(i,j,g,\mathsf{pr}_2(F_i)(a)))} \\
\mathsf{transp}^{x\mapsto\tilde{\xi}(\xi(x))=x}(\mathsf{glue}_{V_2}(\mathsf{inr}(i,j,g,a)),\mathsf{refl}_{\mathsf{inl}(a)}) &= \mathsf{ap}_{\tilde{\xi}}(\mathsf{ap}_{\xi}(\mathsf{glue}(\mathsf{inr}(i,j,g,a))))^{-1}\cdot\mathsf{glue}(\mathsf{inr}(i,j,g,a)) \\
&= \mathsf{glue}(\mathsf{inr}(i,j,g,a))^{-1}\cdot\mathsf{glue}(\mathsf{inr}(i,j,g,a)) \\
&= \mathsf{refl}_{\mathsf{inr}(\mathsf{inr}(i,j,g,\mathsf{pr}_2(F_i)(a)))}
\end{aligned}$$

Conversely, we derive a homotopy $\xi \circ \tilde{\xi} \sim \mathsf{id}_{\bigvee \vee \bigvee}$ by induction on $\bigvee \vee \bigvee$ as follows.

$$\begin{aligned}
\xi(\tilde{\xi}(\mathsf{inl}(\mathsf{inr}(i,j,g,x)))) &\equiv \xi(\mathsf{inr}(\mathsf{inl}(i,j,g,x))) \\
&\equiv \mathsf{inl}(\mathsf{inr}(i,j,g,x)) \\
\xi(\tilde{\xi}(\mathsf{inl}(\mathsf{inl}(a)))) &\equiv \xi(\mathsf{inl}(a)) \\
&\equiv \mathsf{inl}(\mathsf{inl}(a))
\end{aligned}$$

$$\begin{aligned}
&\mathsf{transp}^{x \mapsto \xi(\tilde{\xi}(\mathsf{inl}(x)))=\mathsf{inl}(x)}(\mathsf{glue}_{\bigvee_{i,j,g} \mathsf{pr}_1(F_i)}(i,j,g,a), \mathsf{refl}_{\mathsf{inl}(a)}) \\
&\quad = \mathsf{ap}_{\xi}(\mathsf{ap}_{\underbrace{\tilde{\xi} \circ \mathsf{inl}}_{\epsilon_1}}(\mathsf{glue}(i,j,g,a)))^{-1} \cdot \mathsf{ap}_{\mathsf{inl}}(\mathsf{glue}(i,j,g,a)) \\
&\quad = \mathsf{ap}_{\xi}(\mathsf{glue}_{V_2}(\mathsf{inl}(i,j,g,a)))^{-1} \cdot \mathsf{ap}_{\mathsf{inl}}(\mathsf{glue}(i,j,g,a)) \quad (\mathsf{ap}_{-^{-1} \cdot \mathsf{ap}_{\mathsf{inl}}(\mathsf{glue}(i,j,g,a))}(\mathsf{ap}_{\mathsf{ap}_{\xi}}(\rho_{\epsilon_1}(i,j,g,a)))) \\
&\quad = \mathsf{ap}_{\mathsf{inl}}(\mathsf{glue}(i,j,g,a))^{-1} \cdot \mathsf{ap}_{\mathsf{inl}}(\mathsf{glue}(i,j,g,a)) \quad (\mathsf{ap}_{-^{-1} \cdot \mathsf{ap}_{\mathsf{inl}}(\mathsf{glue}(i,j,g,a))}(\rho_{\xi}(\mathsf{inl}(i,j,g,a)))) \\
&\quad = \mathsf{refl}_{\mathsf{inr}(\mathsf{inl}(i,j,g,\mathsf{pr}_2(F_i)(a)))}
\end{aligned}$$

$$\begin{aligned}
\xi(\tilde{\xi}(\mathsf{inr}(\mathsf{inr}(i,j,g,x)))) &\equiv \xi(\mathsf{inr}(\mathsf{inr}(i,j,g,x))) \\
&\equiv \mathsf{inr}(\mathsf{inr}(i,j,g,x)) \\
\xi(\tilde{\xi}(\mathsf{inr}(\mathsf{inl}(a)))) &\equiv \xi(\mathsf{inl}(a)) \\
&\equiv \mathsf{inl}(\mathsf{inl}(a)) \\
&= \mathsf{inr}(\mathsf{inl}(a)) \qquad (\mathsf{glue}_{\bigvee \vee \bigvee}(a))
\end{aligned}$$

$$\begin{aligned}
&\mathsf{transp}^{x \mapsto \xi(\tilde{\xi}(\mathsf{inr}(x)))=\mathsf{inr}(x)}(\mathsf{glue}_{\bigvee_{i,j,g} \mathsf{pr}_1(F_i)}(i,j,g,a), \mathsf{glue}(a)) \\
&\quad = \mathsf{ap}_{\xi}(\mathsf{ap}_{\underbrace{\tilde{\xi} \circ \mathsf{inr}}_{\epsilon_2}}(\mathsf{glue}(i,j,g,a)))^{-1} \cdot \mathsf{glue}(a) \cdot \mathsf{ap}_{\mathsf{inr}}(\mathsf{glue}(i,j,g,a)) \\
&\quad = \mathsf{ap}_{\xi}(\mathsf{glue}_{V_2}(\mathsf{inr}(i,j,g,a)))^{-1} \cdot \mathsf{glue}(a) \cdot \mathsf{ap}_{\mathsf{inr}}(\mathsf{glue}(i,j,g,a)) \\
&\qquad\qquad (\mathsf{ap}_{-^{-1} \cdot \mathsf{glue}(a) \cdot \mathsf{ap}_{\mathsf{inr}}(\mathsf{glue}(i,j,g,a))}(\mathsf{ap}_{\mathsf{ap}_{\xi}}(\rho_{\epsilon_2}(i,j,g,a)))) \\
&\quad = (\mathsf{glue}(a) \cdot \mathsf{ap}_{\mathsf{inr}}(\mathsf{glue}(i,j,g,a)))^{-1} \cdot \mathsf{glue}(a) \cdot \mathsf{ap}_{\mathsf{inr}}(\mathsf{glue}(i,j,g,a)) \\
&\qquad\qquad (\mathsf{ap}_{-^{-1} \cdot \mathsf{glue}(a) \cdot \mathsf{ap}_{\mathsf{inr}}(\mathsf{glue}(i,j,g,a))}(\rho_{\xi}(\mathsf{inl}(i,j,g,a)))) \\
&\quad = \mathsf{refl}_{\mathsf{inr}(\mathsf{inr}(i,j,g,\mathsf{pr}_2(F_i)(a)))}
\end{aligned}$$

$$\begin{aligned}
&\mathsf{transp}^{x \mapsto \xi(\tilde{\xi}(x))=x}(\mathsf{glue}_{\bigvee \vee \bigvee}(a), \mathsf{refl}_{\mathsf{inl}(\mathsf{inl}(a))}) \\
&\quad = \mathsf{ap}_{\xi}(\mathsf{ap}_{\tilde{\xi}}(\mathsf{glue}(a)))^{-1} \cdot \mathsf{glue}(a) \\
&\quad = \mathsf{refl}_{\mathsf{inl}(\mathsf{inl}(a))} \cdot \mathsf{glue}(a) \\
&\quad \equiv \mathsf{glue}(a)
\end{aligned}$$

$\square$

Now, define $\sigma : \left(\bigvee_{i,j,g} \mathsf{pr}_1(F_i)\right) \vee \left(\bigvee_{i,j,g} \mathsf{pr}_1(F_i)\right) \to \bigvee_i \mathsf{pr}_1(F_i)$ by double induction on pushouts through the commuting square

$$\begin{array}{ccc}
A & \longrightarrow & \bigvee_{i,j,g} \mathsf{pr}_1(F_i) \\
\Big\downarrow & \mathsf{refl}_{\mathsf{inl}(a)} & \Big\downarrow \alpha_2 \\
\bigvee_{i,j,g} \mathsf{pr}_1(F_i) & \xrightarrow[\alpha_1]{} & \bigvee_i \mathsf{pr}_1(F_i)
\end{array}$$

Here, α_1 and α_2 are induced by the commuting squares

$$\begin{array}{ccc}
\left(\sum_{(i,j):\Gamma_0\times\Gamma_0} \Gamma_1(i,j)\right) \times A & \longrightarrow & \sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0} \Gamma_1(i,j)} \mathsf{pr}_1(F_i) \\
\Big\downarrow & \mathsf{glue}_{\bigvee_i \mathsf{pr}_1(F_i)}(i,a) & \Big\downarrow \mathsf{inr}(i,x) \\
A & \xrightarrow[a\mapsto\mathsf{inl}(a)]{} & \bigvee_i \mathsf{pr}_1(F_i)
\end{array}$$

$$\begin{array}{ccc}
\left(\sum_{(i,j):\Gamma_0\times\Gamma_0} \Gamma_1(i,j)\right) \times A & \longrightarrow & \sum_{(i,j,g):\sum_{(i,j):\Gamma_0\times\Gamma_0} \Gamma_1(i,j)} \mathsf{pr}_1(F_i) \\
\Big\downarrow & \mathsf{glue}_{\bigvee_i \mathsf{pr}_1(F_i)}(j,a)\cdot\mathsf{ap}_{\mathsf{inr}(j,-)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} & \Big\downarrow \mathsf{inr}(j,\mathsf{pr}_1(F_{i,j,g})(x)) \\
A & \xrightarrow[a\mapsto\mathsf{inl}(a)]{} & \bigvee_i \mathsf{pr}_1(F_i)
\end{array}$$

respectively. We have a map

$$\begin{array}{ccccc}
V_1 & \xleftarrow{\delta_1} & V_2 & \xrightarrow{\delta_2} & V_3 \\
\Big\| & & \simeq \Big\downarrow \xi & & \Big\| \\
\bigvee_{i,j,g} \mathsf{pr}_1(F_i) & \xleftarrow[\mathsf{id}\vee\mathsf{id}]{} & \left(\bigvee_{i,j,g} \mathsf{pr}_1(F_i)\right) \vee \left(\bigvee_{i,j,g} \mathsf{pr}_1(F_i)\right) & \xrightarrow[\sigma]{} & \bigvee_i \mathsf{pr}_1(F_i)
\end{array}$$

of spans. Denote the pushout of the lower span by PW.

Indeed, we have homotopies

$$\begin{aligned}
(\mathsf{id} \vee \mathsf{id})\,(\xi(\mathsf{inl}(a))) &\equiv \mathsf{inl}(a) \\
&\equiv \delta_1(\mathsf{inl}(a)) \\
(\mathsf{id} \vee \mathsf{id})\,(\xi(\mathsf{inr}(\mathsf{inl}(i,j,g,x)))) &\equiv \mathsf{inr}(i,j,g,x) \\
&\equiv \delta_1(\mathsf{inr}(\mathsf{inl}(i,j,g,x))) \\
(\mathsf{id} \vee \mathsf{id})\,(\xi(\mathsf{inr}(\mathsf{inr}(i,j,g,x)))) &\equiv \mathsf{inr}(i,j,g,x) \\
&\equiv \delta_1(\mathsf{inr}(\mathsf{inr}(i,j,g,x)))
\end{aligned}$$

$$\begin{aligned}
&\mathsf{transp}^{x \mapsto (\mathsf{id} \vee \mathsf{id})(\xi(x)) = \delta_1(x)}(\mathsf{glue}_{V_2}(\mathsf{inl}(i,j,g)), \mathsf{refl}_{\mathsf{inl}(a)}) \\
&\quad = \mathsf{ap}_{\mathsf{id} \vee \mathsf{id}}(\mathsf{ap}_{\xi}(\mathsf{glue}(\mathsf{inl}(i,j,g))))^{-1} \cdot \mathsf{ap}_{\delta_1}(\mathsf{glue}(\mathsf{inl}(i,j,g))) \\
&\quad = \mathsf{ap}_{\underbrace{(\mathsf{id} \vee \mathsf{id}) \circ \mathsf{inl}}_{\mathsf{id}}}(\mathsf{glue}(i,j,g,a))^{-1} \cdot \mathsf{glue}(i,j,g,a) \\
&\quad = \mathsf{refl}_{\mathsf{inr}(i,j,g,\mathsf{pr}_2(F_i)(a))} \\
&\mathsf{transp}^{x \mapsto (\mathsf{id} \vee \mathsf{id})(\xi(x)) = \delta_1(x)}(\mathsf{glue}_{V_2}(\mathsf{inr}(i,j,g)), \mathsf{refl}_{\mathsf{inl}(a)}) \\
&\quad = \mathsf{ap}_{\mathsf{id} \vee \mathsf{id}}(\mathsf{ap}_{\xi}(\mathsf{glue}(\mathsf{inr}(i,j,g))))^{-1} \cdot \mathsf{ap}_{\delta_1}(\mathsf{glue}(\mathsf{inr}(i,j,g))) \\
&\quad = \mathsf{ap}_{\underbrace{(\mathsf{id} \vee \mathsf{id}) \circ \mathsf{inr}}_{\mathsf{id}}}(\mathsf{glue}(i,j,g,a))^{-1} \cdot \mathsf{glue}(i,j,g,a) \\
&\quad = \mathsf{refl}_{\mathsf{inr}(i,j,g,\mathsf{pr}_2(F_i)(a))}
\end{aligned}$$

$$\begin{aligned}
\sigma(\xi(\mathsf{inl}(a))) &\equiv \mathsf{inl}(a) \\
&\equiv \delta_2(\mathsf{inl}(a)) \\
\sigma(\xi(\mathsf{inr}(\mathsf{inl}(i,j,g,x)))) &\equiv \sigma(\mathsf{inl}(\mathsf{inr}(i,j,g,x))) \\
&\equiv \alpha_1(\mathsf{inr}(i,j,g,x)) \\
&\equiv \mathsf{inr}(i,x) \\
&\equiv \delta_2(\mathsf{inr}(\mathsf{inl}(i,j,g,x))) \\
\sigma(\xi(\mathsf{inr}(\mathsf{inr}(i,j,g,x)))) &\equiv \sigma(\mathsf{inr}(\mathsf{inr}(i,j,g,x))) \\
&\equiv \alpha_2(\mathsf{inr}(i,j,g,x)) \\
&\equiv \mathsf{inr}(j, \mathsf{pr}_1(F_{i,j,g})(x)) \\
&\equiv \delta_2(\mathsf{inr}(\mathsf{inr}(i,j,g,x)))
\end{aligned}$$

$$
\begin{aligned}
&\mathsf{transp}^{x\mapsto\sigma(\xi(x))=\delta_2(x)}(\mathsf{glue}_{V_2}(\mathsf{inl}(i,j,g)),\mathsf{refl}_{\mathsf{inl}(a)})\\
&\quad=\mathsf{ap}_\sigma(\mathsf{ap}_\xi(\mathsf{glue}(\mathsf{inl}(i,j,g))))^{-1}\cdot\mathsf{ap}_{\delta_2}(\mathsf{glue}(\mathsf{inl}(i,j,g)))\\
&\quad=\mathsf{ap}_{\underbrace{\sigma\,\circ\,\mathsf{inl}}_{\alpha_1}}(\mathsf{glue}(i,j,g,a))^{-1}\cdot\mathsf{glue}(i,a)\\
&\quad=\mathsf{glue}(i,a)^{-1}\cdot\mathsf{glue}(i,a)\\
&\quad=\mathsf{refl}_{\mathsf{inr}(i,\mathsf{pr}_2(F_i)(a))}\\
&\mathsf{transp}^{x\mapsto\sigma(\xi(x))=\delta_2(x)}(\mathsf{glue}_{V_2}(\mathsf{inr}(i,j,g)),\mathsf{refl}_{\mathsf{inl}(a)})\\
&\quad=\mathsf{ap}_\sigma(\mathsf{ap}_\xi(\mathsf{glue}(\mathsf{inr}(i,j,g))))^{-1}\cdot\mathsf{ap}_{\delta_2}(\mathsf{glue}(\mathsf{inr}(i,j,g)))\\
&\quad=\mathsf{ap}_{\underbrace{\sigma\,\circ\,\mathsf{inr}}_{\alpha_2}}(\mathsf{glue}(i,j,g,a))^{-1}\cdot\mathsf{glue}(j,a)\cdot\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{(j,-)}(\mathsf{pr}_2(F_{i,j,g})(a)))\\
&\quad=\Big(\mathsf{glue}(j,a)\cdot\mathsf{ap}_{\mathsf{inr}(j,-)}(\mathsf{pr}_2(F_{i,j,g})(a))\Big)^{-1}\cdot\mathsf{glue}(j,a)\cdot\mathsf{ap}_{\mathsf{inr}}(\mathsf{ap}_{(j,-)}(\mathsf{pr}_2(F_{i,j,g})(a)))\\
&\quad=\mathsf{refl}_{\mathsf{inr}(j,\mathsf{pr}_2(F_j)(a))}
\end{aligned}
$$

This induces an equivalence $\tau_2 : P_V \xrightarrow{\simeq} \mathsf{PW}$ of pushouts.

Lemma 5.5.2. *We have an equivalence*

$$
\begin{array}{ccc}
\mathsf{colim}_\Gamma\, A & \xrightarrow{\ \psi\ } & \mathsf{colim}_\Gamma(\mathcal{F}(F))\\
{\scriptstyle w_0}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\big\downarrow{\scriptstyle w_1}\\
H_2 & \xrightarrow[\ \eta_1\ ]{} & H_1
\end{array}
\qquad (\psi\text{-}\eta_1\text{-}\mathsf{sq})
$$

between ψ *and* η_1.

Proof. Define w_0 and w_1 by the following cocones under A and $\mathcal{F}(F)$, respectively.

$$
\begin{array}{ccc}
A & \xrightarrow{\ \mathsf{id}_A\ } & A\\
& {\scriptstyle \mathsf{inr}(i,-)}\searrow \quad \swarrow{\scriptstyle \mathsf{inr}(j,-)} & \\
& H_2 &
\end{array}
\qquad (a\mapsto \mathsf{glue}_{H_2}(\mathsf{inr}(i,j,g,a))^{-1}\cdot\mathsf{glue}_{H_2}(\mathsf{inl}(i,j,g,a)))
$$

$$
\begin{array}{ccc}
\mathsf{pr}_1(F_i) & \xrightarrow{\ \mathsf{pr}_1(F_{i,j,g})\ } & \mathsf{pr}_1(F_j)\\
& {\scriptstyle \mathsf{inr}(i,-)}\searrow \quad \swarrow{\scriptstyle \mathsf{inr}(j,-)} & \\
& H_1 &
\end{array}
\qquad (x\mapsto \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1}\cdot\mathsf{glue}_{H_1}(\mathsf{inl}(i,j,g,x)))
$$

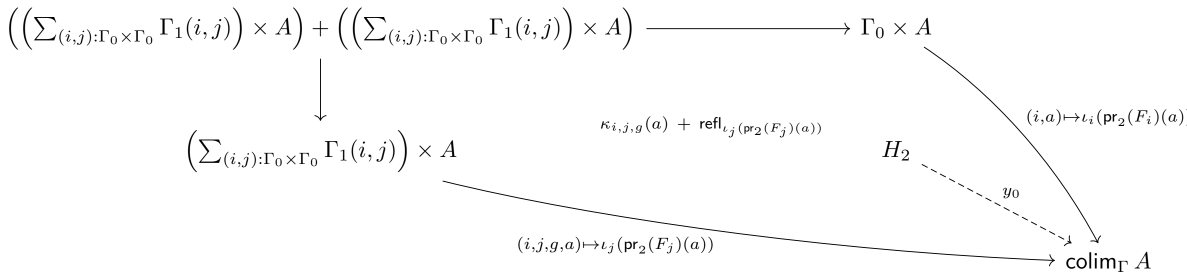
To see that the square (ψ-η_1-$\mathtt{sq}$) commutes, proceed by induction on $\mathsf{colim}_\Gamma A$. Note that

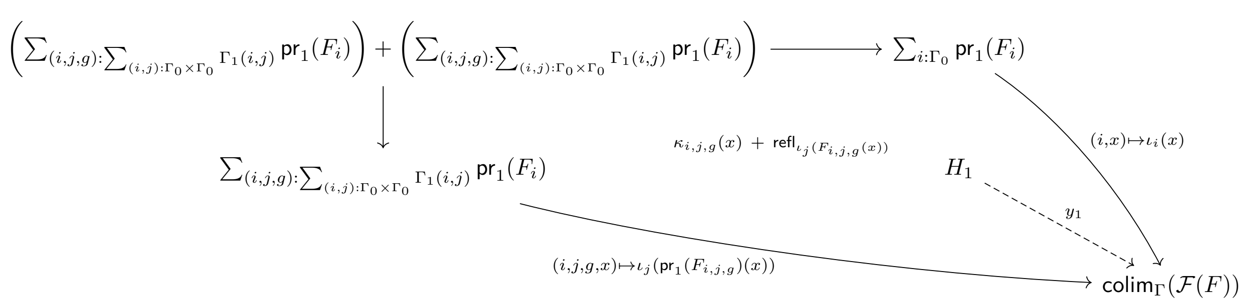
$$\begin{aligned} & \eta_1(w_0(\iota_i(a))) \\ \equiv\ & \eta_1(\mathsf{inr}(i,a)) \\ \equiv\ & \mathsf{inr}(i, \mathsf{pr}_2(F_i)(a)) \\ \equiv\ & w_1(\iota_i(\mathsf{pr}_2(F_i)(a))) \\ \equiv\ & w_1(\psi(\iota_i(a))) \end{aligned}$$

and

$$\begin{gathered} \mathsf{transp}^{x \mapsto \eta_1(w_0(x)) = w_1(\psi(x))}(\kappa_{i,j,g}(a), \mathsf{refl}_{\mathsf{inr}(j,\mathsf{pr}_2(F_j)(a))}) \\ \| \\ \mathsf{ap}_{\eta_1}(\mathsf{ap}_{w_0}(\kappa_{i,j,g}(a)))^{-1} \cdot \mathsf{ap}_{w_1}(\mathsf{ap}_\psi(\kappa_{i,j,g}(a))) \\ \| \\ \mathsf{ap}_{\eta_1}(\mathsf{glue}(\mathsf{inl}(i,j,g,a)))^{-1} \cdot \mathsf{ap}_{\eta_1}(\mathsf{glue}(\mathsf{inr}(i,j,g,a))) \cdot \mathsf{ap}_{w_1 \circ \iota_j}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \mathsf{ap}_{w_1}(\kappa_{i,j,g}(\mathsf{pr}_2(F_i)(a))) \\ \| \\ \mathsf{glue}_{H_1}(\mathsf{inl}(i,j,g,\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,\mathsf{pr}_2(F_i)(a))) \cdot \mathsf{ap}_{\mathsf{inr}(j,-)}(\mathsf{pr}_2(F_{i,j,g})(a)) \cdot \\ \mathsf{ap}_{\mathsf{inr}(j,-)}(\mathsf{pr}_2(F_{i,j,g})(a))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,\mathsf{pr}_2(F_i)(a)))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inl}(i,j,g,\mathsf{pr}_2(F_i)(a))) \\ \| \\ \mathsf{refl}_{\mathsf{inr}(i,\mathsf{pr}_2(F_i)(a))} \end{gathered}$$

Next, define quasi-inverses y_0 and y_1 of w_0 and w_1, respectively, by recursion on puhsouts:





On the one hand,

$$
\begin{aligned}
&y_1(w_1(\iota_i(x))) \equiv y_1(\mathsf{inr}(i,x)) \equiv \iota_i(x) \\
&\mathsf{transp}^{z\mapsto y_1(w_1(z))=z}(\kappa_{i,j,g}(x), \mathsf{refl}_{\iota_j(\mathsf{pr}_1(F_{i,j,g})(x))}) \\
&\quad = \mathsf{ap}_{y_1}(\mathsf{ap}_{w_1}(\kappa_{i,j,g}(x)))^{-1} \cdot \kappa_{i,j,g}(x) \\
&\quad = \mathsf{ap}_{y_1}(\mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inl}(i,j,g,x)))^{-1} \cdot \kappa_{i,j,g}(x) \\
&\quad = \kappa_{i,j,g}(x)^{-1} \cdot \kappa_{i,j,g}(x) \\
&\quad = \mathsf{refl}_{\iota_i(x)}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
w_1(y_1(\mathsf{inl}(i,j,g,x))) &\equiv w_1(\iota_j(\mathsf{pr}_1(F_{i,j,g})(x))) \\
&\equiv \mathsf{inr}(j, \mathsf{pr}_1(F_{i,j,g})(x)) \\
&= \mathsf{inl}(i,j,g,x) && (\mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1}) \\
w_1(y_1(\mathsf{inr}(i,x))) &\equiv w_1(\iota_i(x)) \equiv \mathsf{inr}(i,x)
\end{aligned}
$$

$$
\begin{aligned}
&\mathsf{transp}^{z\mapsto w_1(y_1(z))=z}(\mathsf{glue}_{H_1}(\mathsf{inl}(i,j,g,x)), \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1}) \\
&\quad = \mathsf{ap}_{w_1}(\mathsf{ap}_{y_1}(\mathsf{glue}_{H_1}(\mathsf{inl}(i,j,g,x))))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inl}(i,j,g,x)) \\
&\quad = \mathsf{ap}_{w_1}(\kappa_{i,j,g}(x))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inl}(i,j,g,x)) \\
&\quad = \left(\mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inl}(i,j,g,x))\right)^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inl}(i,j,g,x)) \\
&\quad = \mathsf{refl}_{\mathsf{inr}(i,x)} \\
&\mathsf{transp}^{z\mapsto w_1(y_1(z))=z}(\mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x)), \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1}) \\
&\quad = \mathsf{ap}_{w_1}(\mathsf{ap}_{y_1}(\mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x)) \\
&\quad = \mathsf{ap}_{w_1}(\mathsf{refl}_{\iota_j(\mathsf{pr}_1(F_{i,j,g})(x))})^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x))^{-1} \cdot \mathsf{glue}_{H_1}(\mathsf{inr}(i,j,g,x)) \\
&\quad = \mathsf{refl}_{\mathsf{inr}(j,\mathsf{pr}_1(F_{i,j,g})(x))}
\end{aligned}
$$

Likewise, we find that $y_0 \circ w_0 \sim \mathsf{id}_{\mathsf{colim}_\Gamma A}$ and $w_0 \circ y_0 \sim \mathsf{id}_{H_2}$. □

The map (ψ-η_1-`sq`) fits into an equivalence of spans

$$
\begin{array}{ccccc}
A & \xleftarrow{[\mathsf{id}_A]} & \mathsf{colim}_\Gamma A & \xrightarrow{\psi} & \mathsf{colim}_\Gamma(\mathcal{F}(F)) \\
{\scriptstyle \mathsf{inl}}\downarrow{\scriptstyle \simeq} & \mathsf{glue}_{H_3}(a)^{-1} & {\scriptstyle w_0}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle w_1} \\
H_3 & \xleftarrow[\eta_2]{} & H_2 & \xrightarrow[\eta_1]{} & H_1
\end{array}
$$

which gives us an equivalence $\tau_0 : \mathsf{colim}^A_\Gamma F \xrightarrow{\simeq} P_H$ of pushouts.

Corollary 5.5.3. *We have an equivalence* $T_F : \mathsf{colim}^A_\Gamma F \xrightarrow{\simeq} \mathsf{PW}$ *such that*

$$T_F(\mathsf{inl}(a)) \equiv \mathsf{inl}(\mathsf{inl}(a))$$
$$T_F(\mathsf{inr}(\iota_i(x))) \equiv \mathsf{inr}(\mathsf{inr}(i, x))$$

Proof. Take $T_F :\equiv \tau_2 \circ \tau_1 \circ \tau_0$. □

6 Universality of colimits

Let $A : \mathcal{U}$. Let Γ be a graph and F be an A-diagram over Γ. We say that $\mathsf{colim}^A_\Gamma(F)$ is *universal*, or *pullback-stable*, if for every pullback square

$$\begin{array}{ccc} \mathsf{colim}^A_\Gamma(F) \times_V Y & \xrightarrow{\pi_2} & Y \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle h} \\ \mathsf{colim}^A_\Gamma(F) & \xrightarrow[f]{} & V \end{array} \tag{pb}$$

in $A/\mathcal{U}$, the canonical map

$$\sigma_{f,h} \ : \ \mathsf{colim}^A_{i:\Gamma}(F_i \times_V Y) \to_A \mathsf{colim}^A_\Gamma(F) \times_V Y$$

is an equivalence.

Lemma 6.0.1. *The forgetful functor* $\mathcal{F} : A/\mathcal{U} \to \mathcal{U}$ *preserves limits.*

Proof. Consider the free functor $((-) + A) : \mathcal{U} \to A/\mathcal{U}$. This is left adjoint to $\mathcal{F}$. Let F be an A-diagram over a graph Γ. Let (C, r, K) be a limiting A-cone over F. For each $X : \mathcal{U}$, it's easy to check that the composite of equivalences

$$\begin{aligned} & X \to \mathsf{pr}_1(C) \\ \simeq\ & (X + A) \to_A C \\ \simeq\ & \mathsf{lim}_{i:\Gamma}((X + A) \to_A F_i) \\ \simeq\ & \mathsf{lim}_{i:\Gamma}(X \to \mathsf{pr}_1(F_i)) \end{aligned}$$

equals the function $(\mathcal{F}(C, r, K) \circ -)$, which is thus an equivalence. □

Theorem 6.0.2. *All colimits in* $\mathcal{U}$ *are universal.*

We have formalized Theorem 6.0.2 in Agda (see the folder [7, `Pullback-stability`]).

Corollary 6.0.3. *For each tree* Γ *and each* A*-diagram* F *over* Γ*, the colimit* $\mathsf{colim}^A_\Gamma(F)$ *is universal.*

Proof. Suppose that Γ is a tree and consider the pullback square (pb). By Corollary 5.4.6 combined with Theorem 6.0.2, the function

$$\mathsf{pr}_1(\mathsf{colim}^A_{i:\Gamma}(F_i \times_V Y)) \xrightarrow{\mathsf{pr}_1(\sigma_{f,h})} \mathsf{pr}_1(\mathsf{colim}^A_\Gamma(F)) \times_{\mathsf{pr}_1(V)} \mathsf{pr}_1(Y)$$

is an equivalence. The codomain is in this form because $\mathcal{F}$ preserves pullbacks by Lemma 6.0.1. It follows that $\sigma_{f,h}$ is an equivalence. □

Note 6.0.4. We can construct pullbacks in $A/\mathcal{U}$ as follows. Consider a cospan S

$$\begin{array}{ccc} & & Y \\ & & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

in $A/\mathcal{U}$ and form the standard pullback

$$\mathsf{pr}_1(X) \times_{\mathsf{pr}_1(Z)} \mathsf{pr}_1(Y) \ :\equiv \sum_{x:\mathsf{pr}_1(X)} \sum_{y:\mathsf{pr}_1(Y)} \mathsf{pr}_1(f)(x) = \mathsf{pr}_1(g)(y)$$

of $\mathcal{F}(S)$ in $\mathcal{U}$ [2, Definition 4.1.1]. Define $\mu_{f,g} : A \to \mathsf{pr}_1(X) \times_{\mathsf{pr}_1(Z)} \mathsf{pr}_1(Y)$ by

$$a \ \mapsto \ \big(\mathsf{pr}_2(X)(a), \mathsf{pr}_2(Y)(a), \mathsf{pr}_2(f)(a) \cdot \mathsf{pr}_2(g)(a)^{-1}\big)$$

Now we have a cone

$$\begin{array}{ccc} \overbrace{\big(\mathsf{pr}_1(X) \times_{\mathsf{pr}_1(Z)} \mathsf{pr}_1(Y), \mu_{f,g}\big)}^{\Phi} & \xrightarrow{(\pi_y, \mathsf{refl}_x)} & Y \\ {\scriptstyle (\pi_x, \mathsf{refl}_y)}\big\downarrow & {\scriptstyle \langle (x,y,p) \mapsto p, H_p \rangle} & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array} \tag{sq}$$

over S where $H_p(a)$ denotes the evident path $\big(\mathsf{pr}_2(f)(a) \cdot \mathsf{pr}_2(g)(a)^{-1}\big)^{-1} \cdot \mathsf{pr}_2(f)(a) = \mathsf{pr}_2(g)(a)$ for each $a : A$. We have used π to denote field projection for a Σ-type. We claim that (sq) is a pullback square, i.e., the function

$$(\mathsf{sq} \circ -) \ : \ ((T, f_T) \to_A \Phi) \to \mathsf{Cone}((T, f_T)\,; f, g)$$

is an equivalence for each $(T, f_T) : A/\mathcal{U}$. Indeed, for all $K :\equiv (k_1, k_2, \langle q, Q \rangle) : \mathsf{Cone}((T, f_T)\,; f, g)$, the fiber $\mathsf{fib}_{(\mathsf{sq}\circ -)}(K)$ is equivalent to the type of data

$$\begin{aligned}
d &: T \to \mathsf{pr}_1(X) \times_{\mathsf{pr}_1(Z)} \mathsf{pr}_1(Y) \\
h_1 &: \pi_x \circ d \sim \mathsf{pr}_1(k_1) \\
h_2 &: \pi_y \circ d \sim \mathsf{pr}_1(k_2) \\
\tau &: \prod_{t:T} \mathsf{ap}_{\mathsf{pr}_1(f)}(h_1(t)) \cdot q(t) \cdot \mathsf{ap}_{\mathsf{pr}_1(g)}(h_2(t))^{-1} = \pi_p(d(t))
\end{aligned}$$

$$\begin{aligned}
d_p &: d \circ f_T \sim \mu_{f,g} \\
H_1 &: \prod_{a:A} h_1(f_T(a))^{-1} \cdot \mathsf{ap}_{\pi_x}(d_p(a)) = \mathsf{pr}_2(k_1)(a) \\
H_2 &: \prod_{a:A} h_2(f_T(a))^{-1} \cdot \mathsf{ap}_{\pi_y}(d_p(a)) = \mathsf{pr}_2(k_2)(a) \\
\nu &: \prod_{a:A} \Theta(\tau, d_p, H_1, H_2, a) = Q(a)
\end{aligned}$$

where $\Theta(\tau, d_p, H_1, H_2, a)$ denotes the chain of paths

$$\begin{aligned}
& q(f_T(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{pr}_2(k_1)(a)) \cdot \mathsf{pr}_2(f)(a) \\
&= \quad \text{via } \tau(f_T(a)) \\
& \mathsf{ap}_{\mathsf{pr}_1(g)}(h_2(f_T(a)))^{-1} \cdot \pi_p(d(f_T(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(h_1(f_T(a))) \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{pr}_2(k_1)(a)) \cdot \mathsf{pr}_2(f)(a) \\
&= \quad \text{via } H_1(a) \text{ and } H_2(a) \\
& \mathsf{ap}_{\mathsf{pr}_1(g)}(\mathsf{pr}_2(k_2)(a) \cdot \mathsf{ap}_{\pi_y}(d_p(a))^{-1}) \cdot \pi_p(d(f_T(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{ap}_{\pi_x}(d_p(a)) \cdot \mathsf{pr}_2(k_1)(a)^{-1}) \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{pr}_2(k_1)(a)) \cdot \mathsf{pr}_2(f)(a) \\
&= \\
& \mathsf{ap}_{\mathsf{pr}_1(g)}(\mathsf{pr}_2(k_2)(a)) \cdot \mathsf{ap}_{\mathsf{pr}_1(g)}(\mathsf{ap}_{\pi_y}(d_p(a))^{-1}) \cdot \pi_p(d(f_T(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{ap}_{\pi_x}(d_p(a))) \cdot \mathsf{pr}_2(f)(a) \\
&= \quad \text{via } d_p(a) \\
& \mathsf{ap}_{\mathsf{pr}_1(g)}(\mathsf{pr}_2(k_2)(a)) \cdot \mathsf{ap}_{\mathsf{pr}_1(g)}(\mathsf{ap}_{\pi_y}(d_p(a))^{-1}) \cdot \mathsf{ap}_{\mathsf{pr}_1(g)}(\mathsf{ap}_{\pi_y}(d_p(a))) \cdot \\
& \quad \left(\mathsf{pr}_2(f)(a) \cdot \mathsf{pr}_2(g)(a)^{-1}\right)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{ap}_{\pi_x}(d_p(a)))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{ap}_{\pi_x}(d_p(a))) \cdot \mathsf{pr}_2(f)(a) \\
&= \\
& \mathsf{ap}_{\mathsf{pr}_1(g)}(\mathsf{pr}_2(k_2)(a)) \cdot \mathsf{pr}_2(g)(a)
\end{aligned}$$

We can contract the four left-hand fields to the point defined by

$$\begin{aligned}
d(t) &:= (\mathsf{pr}_1(k_1)(t), \mathsf{pr}_1(k_2)(t), q(t)) \\
h_1(t) &:= \mathsf{refl}_{\mathsf{pr}_1(k_1)(t)} \\
h_2(t) &:= \mathsf{refl}_{\mathsf{pr}_1(k_2)(t)} \\
\tau(t) &:= \mathsf{RId}(q(t))
\end{aligned}$$

because $\mathsf{pr}_1(X) \times_{\mathsf{pr}_1(Z)} \mathsf{pr}_1(Y)$ is the standard pullback of $\mathcal{F}(S)$. Therefore, it suffices to prove that the type of data

$$\begin{aligned}
d_1 &: \mathsf{pr}_1(k_1) \circ f_T \sim \mathsf{pr}_2(X) \\
H_1 &: d_1 \sim \mathsf{pr}_2(k_1) \\
d_2 &: \mathsf{pr}_2(k_2) \circ f_T \sim \mathsf{pr}_2(Y) \\
H_2 &: d_2 \sim \mathsf{pr}_2(k_2) \\
d_3 &: \prod_{a:A} q(f_T(a)) = \mathsf{ap}_{\mathsf{pr}_1(f)}(d_1(a)) \cdot \mathsf{pr}_2(f)(a) \cdot \mathsf{pr}_2(g)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(g)}(d_2(a))^{-1} \\
\nu &: \prod_{a:A} \Theta(\tau, d_3, H_1, H_2, a) = Q(a)
\end{aligned}$$

is contractible. We can contract the first four fields to $(\mathsf{pr}_2(k_1), \mathsf{refl}, \mathsf{pr}_2(k_2), \mathsf{refl})$. This leaves us with the type of data

$$\begin{aligned}
\nu_1 &: \prod_{a:A} q(f_T(a))^{-1} = \mathsf{ap}_{\mathsf{pr}_1(g)}(\mathsf{pr}_2(k_2)(a)) \cdot \mathsf{pr}_2(g)(a) \cdot \mathsf{pr}_2(f)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{pr}_2(k_1)(a))^{-1} \\
\nu_2 &: \prod_{a:A} \Psi(\nu_1, a) = Q(a)
\end{aligned}$$

where $\Psi(\nu_1, a)$ denotes the chain

$$q(f_T(a))^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{pr}_2(k_1)(a)) \cdot \mathsf{pr}_2(f)(a)$$
$$\Big\| \ \mathsf{ap}_{- \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{pr}_2(k_1)(a)) \cdot \mathsf{pr}_2(f)(a)}(\nu_1)$$
$$\left(\mathsf{ap}_{\mathsf{pr}_1(g)}(\mathsf{pr}_2(k_2)(a)) \cdot \mathsf{pr}_2(g)(a) \cdot \mathsf{pr}_2(f)(a)^{-1} \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{pr}_2(k_1)(a))^{-1}\right) \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{pr}_2(k_1)(a)) \cdot \mathsf{pr}_2(f)(a)$$
$$\Big\|$$
$$\mathsf{ap}_{\mathsf{pr}_1(g)}(\mathsf{pr}_2(k_2)(a)) \cdot \mathsf{pr}_2(g)(a)$$

As the function $- \cdot \mathsf{ap}_{\mathsf{pr}_1(f)}(\mathsf{pr}_2(k_1)(a)) \cdot \mathsf{pr}_2(f)(a)$ is an equivalence, it follows that this type is contractible, as desired.

Remark. The category $A/\mathcal{U}$ is usually *not* LCC. Indeed, it is not LCC whenever A is connected. In this case, suppose, for example, that Γ is the discrete graph on $\mathbf{2}$ and define the A-diagram F over Γ by $F_i :\equiv (A, \mathsf{id}_A)$ for each $i : \mathbf{2}$. Then

$$\mathsf{pr}_1(\mathsf{colim}^A_\Gamma(F) \times_{\mathbf{1}} \mathbf{2}) \simeq A + A \not\simeq A + A + A \simeq \mathsf{pr}_1(\mathsf{colim}^A_{i:\Gamma}(F_i \times_{\mathbf{1}} \mathbf{2}))$$

where $A \to \mathbf{2}$ is defined by, say, $a \mapsto 0$.

By the classical adjoint functor theorem, a locally presentable ∞-category is LCC if and only if all its colimits are universal. In this light, Corollary 6.0.3 may be seen as a lower bound on how close $A/\mathcal{U}$ is to being LCC.

7 Coslice colimits preserve connected maps

Let Γ be a graph. Consider the wild category $\mathcal{D}_\Gamma$ of all diagrams over Γ valued in $\mathcal{U}$. Its object type is $\sum_{F:\Gamma_0\to\mathcal{U}}\prod_{i,j:\Gamma_0}\Gamma_1(i,j)\to F_i\to F_j$, and the type of morphisms from F to G is $F\Rightarrow G$. The identity transformation is

$$\mathsf{id}_F \;:=\; \big(\lambda i.\,\mathsf{id}_{F_i}, \lambda i\lambda j\lambda g\lambda x.\mathsf{refl}_{F_{i,j,g}(x)}\big)$$

and the composition of natural transformations is the function

$$\begin{aligned}\circ \;&:\; (G\Rightarrow H)\to(F\Rightarrow G)\to(F\Rightarrow H)\\ (\beta,q)\circ(\alpha,p) \;&:=\; (\lambda i.\beta_i\circ\alpha_i, \lambda i\lambda j\lambda g\lambda x.\,\underbrace{q_{i,j,g}(\alpha(x))\cdot\mathsf{ap}_{\beta_j}(p_{i,j,g}(x))}_{(q*p)(i,j,g,x)}).\end{aligned}$$

Lemma 7.0.1. *Consider a family of types*

$$P \;:\; \begin{array}{l}\prod_{\alpha,\beta:\prod_{i:\Gamma_0}F_i\to G_i}\prod_{p,q:\prod_{i,j,g}G_{i,j,g}\circ\alpha_i\sim\alpha_j\circ F_{i,j,g}}\\ \quad\sum_{W:\prod_{i:\Gamma_0}\alpha_i\sim\beta_i}\Big(\prod_{i,j,g}\prod_{x:F_i}p_{i,j,g}(x)=\mathsf{ap}_{G_{i,j,g}}(W_i(x))\cdot q_{i,j,g}(x)\cdot W_j(F_{i,j,g}(x))^{-1}\Big)\end{array} \;\to\; \mathcal{U}$$

Suppose that for each $(\alpha,p):F\Rightarrow G$, we have a term

$$c(\alpha,p) \;:\; P(\alpha,\alpha,p,p,\lambda i\lambda x.\mathsf{refl}_{\alpha_i(x)},\lambda i\lambda j\lambda g\lambda x.\,\mathsf{RId}(p_{i,j,g}(x))^{-1}).$$

Then we have a section s of P along with an equality $s(\alpha,\alpha,p,p,\lambda i\lambda x.\mathsf{refl}_{\alpha_i(x)},\lambda i\lambda j\lambda g\lambda x.\,\mathsf{RId}(p_{i,j,g}(x))^{-1}) = c(\alpha,p)$ for each $(\alpha,p):F\Rightarrow G$.

Proof. For all $(\alpha,p),(\beta,q):F\Rightarrow G$, we can use Theorem A.0.3 to find an equivalence happly_Γ

$$\begin{gathered}(\alpha,p)=(\beta,q)\\ \wr\!\mid\\ \underbrace{\sum_{W:\prod_{i:\Gamma_0}\alpha_i\sim\beta_i}\prod_{i,j:\Gamma_0}\prod_{g:\Gamma_1(i,j)}\prod_{x:F_i}p_{i,j,g}(x)=\mathsf{ap}_{G_{i,j,g}}(W_i(x))\cdot q_{i,j,g}(x)\cdot W_j(F_{i,j,g}(x))^{-1}}_{(\alpha,p)\sim_\Gamma(\beta,q)}\end{gathered}$$

Thus, the family

$$(\beta,q) \;\mapsto\; (\alpha,p)\sim_\Gamma(\beta,q)$$

pointed by $\big(\mathsf{refl}_{\alpha_i(x)},\mathsf{RId}(p_{i,j,g}(x))^{-1}\big)$ is an identity system on $(F\Rightarrow G,(\alpha,p))$. Now Theorem A.0.2 gives us our desired section. ☐

Notation.

- Define $\langle W,C\rangle :\equiv \mathsf{happly}_\Gamma^{-1}(W,C)$.
- Variables of the form $\alpha^*:F\Rightarrow G$ are abbreviations for pairs (α,α_p).

Lemma 7.0.2. *Let $(\alpha, p), (\beta, q) : F \Rightarrow G$. For all $(W_1, K_1), (W_2, K_2) : (a, p) \sim_\Gamma (\beta, q)$, the type $(W_1, K_1) = (W_2, K_2)$ is equivalent to the type of families of homotopies $H : W_1 \sim W_2$ equipped with a commuting triangle*

$$\begin{array}{ccc} & & \mathsf{ap}_{G_{i,j,g}}(W_2(i,x)) \cdot q_{i,j,g}(x) \cdot W_2(j, F_{i,j,g}(x))^{-1} \\ & \overset{K_2(i,j,g,x)}{\diagup\!\!\diagup} & \Big\| \; {\scriptstyle via\ H(i,x)\ and\ H(j, F_{i,j,g}(x))} \\ p_{i,j,g}(x) & \underset{K_1(i,j,g,x)}{=\!=\!=} & \mathsf{ap}_{G_{i,j,g}}(W_1(i,x)) \cdot q_{i,j,g}(x) \cdot W_1(j, F_{i,j,g}(x))^{-1} \end{array}$$

for all $i, j : \Gamma_0$, $g : \Gamma_1(i,j)$¡ and $x : F_i$.

Proof. By Theorem A.0.3. □

Propositions 7.0.3 to 7.0.5 are easy to verify with Lemma 7.0.1.

Proposition 7.0.3 (Precomposition). *Let $\alpha^*, \beta^* : G \Rightarrow H$ and $\zeta^* : F \Rightarrow G$. For every $(W, C) : \alpha^* \sim_\Gamma \beta^*$, we have a path*

$$\mathsf{ap}_{-\circ\zeta^*}(\langle W, C \rangle) \;=\; \langle \lambda i \lambda x. W_i(\zeta_i(x)), \lambda i \lambda j \lambda g \lambda x. \tau_{W,C}(i,j,g,x) \rangle$$

between elements of the identity type

$$\begin{array}{ccc} F_i & \xrightarrow{F_{i,j,g}} & F_j \\ {\scriptstyle \alpha_i \circ \zeta_i}\downarrow & (\alpha_p * \zeta_p)_{i,j,g} & \downarrow{\scriptstyle \alpha_j \circ \zeta_j} \\ H_i & \xrightarrow[H_{i,j,g}]{} & H_j \end{array} \quad = \quad \begin{array}{ccc} F_i & \xrightarrow{F_{i,j,g}} & F_j \\ {\scriptstyle \beta_i \circ \zeta_i}\downarrow & (\beta_p * \zeta_p)_{i,j,g} & \downarrow{\scriptstyle \beta_j \circ \zeta_j} \\ H_i & \xrightarrow[H_{i,j,g}]{} & H_j \end{array}$$

where $\tau_{W,C}(i,j,g,x)$ denote the chain of paths

$$\begin{gathered} \alpha_p(i,j,g,\zeta_i(x)) \cdot \mathsf{ap}_{\alpha_j}(\zeta_p(i,j,g,x)) \\ \| \\ {\scriptstyle via\ C_{i,j,g}(\zeta_i(x))} \\ \| \\ \left(\mathsf{ap}_{H_{i,j,g}}(W_i(\zeta_i(x))) \cdot \beta_p(i,j,g,\zeta_i(x)) \cdot W_j(G_{i,j,g}(\zeta_i(x)))^{-1}\right) \cdot \mathsf{ap}_{\alpha_j}(\zeta_p(i,j,g,x)) \\ \| \\ {\scriptstyle homotopy\ naturality\ of\ W_j\ at\ \zeta_p(i,j,g,x)} \\ \| \\ \left(\mathsf{ap}_{H_{i,j,g}}(W_i(\zeta_i(x))) \cdot \beta_p(i,j,g,\zeta_i(x)) \cdot \mathsf{ap}_{\beta_j}(\zeta_p(i,j,g,x)) \cdot W_j(\zeta_j(F_{i,j,g}(x)))^{-1} \cdot \mathsf{ap}_{\alpha_j}(\zeta_p(i,j,g,x))^{-1}\right) \cdot \mathsf{ap}_{\alpha_j}(\zeta_p(i,j,g,x)) \\ \Big\| \\ \mathsf{ap}_{H_{i,j,g}}(W_i(\zeta_i(x))) \cdot \left(\beta_p(i,j,g,\zeta_i(x)) \cdot \mathsf{ap}_{\beta_j}(\zeta_p(i,j,g,x))\right) \cdot W_j(\zeta_j(F_{i,j,g}(x)))^{-1} \end{gathered}$$

Proposition 7.0.4 (Postcomposition). *Let $\zeta^* : G \Rightarrow H$. Let $\alpha^*, \beta^* : F \Rightarrow G$. For every*

$(W,C) : \alpha^* \sim_\Gamma \beta^*$, *we have a path*

$$\mathsf{ap}_{\zeta^* \circ -}(\langle W, C\rangle) \;=\; \Big\langle \lambda i \lambda x.\, \mathsf{ap}_{\zeta_i}(W_i(x)), \lambda i \lambda j \lambda g \lambda x. \tau_{W,C}(i,j,g,x) \Big\rangle$$

between elements of the identity type

$$\begin{array}{ccc} F_i & \xrightarrow{F_{i,j,g}} & F_j \\ {\scriptstyle \zeta_i \circ \alpha_i} \big\downarrow & (\zeta_p * \alpha_p)_{i,j,g} & \big\downarrow {\scriptstyle \zeta_j \circ \alpha_j} \\ H_i & \xrightarrow[H_{i,j,g}]{} & H_j \end{array} \quad = \quad \begin{array}{ccc} F_i & \xrightarrow{F_{i,j,g}} & F_j \\ {\scriptstyle \zeta_i \circ \beta_i} \big\downarrow & (\zeta_p * \beta_p)_{i,j,g} & \big\downarrow {\scriptstyle \zeta_j \circ \beta_j} \\ H_i & \xrightarrow[H_{i,j,g}]{} & H_j \end{array}$$

where $\tau_{W,C}(i,j,g,x)$ denotes the chain of equalities

$$\begin{aligned}
&\zeta_p(i,j,g,\alpha_i(x)) \cdot \mathsf{ap}_{\zeta_j}(\alpha_p(i,j,g,x)) \\
&= \text{ via } C_{i,j,g}(x) \\
&\zeta_p(i,j,g,\alpha_i(x)) \cdot \mathsf{ap}_{\zeta_j}(\mathsf{ap}_{G_{i,j,g}}(W_i(x)) \cdot \beta_p(i,j,g,x) \cdot W_j(F_{i,j,g}(x))^{-1}) \\
&= \\
&\zeta_p(i,j,g,\alpha_i(x)) \cdot \mathsf{ap}_{\zeta_j \circ G_{i,j,g}}(W_i(x)) \cdot \mathsf{ap}_{\zeta_j}(\beta_p(i,j,g,x)) \cdot \mathsf{ap}_{\zeta_j}(W_j(F_{i,j,g}(x)))^{-1} \\
&= \text{ homotopy naturality of } \zeta_p \text{ at } W_i(x) \\
&\zeta_p(i,j,g,\alpha_i(x)) \cdot \Big(\zeta_p(i,j,g,\alpha_i(x))^{-1} \cdot \mathsf{ap}_{H_{i,j,g}}(\mathsf{ap}_{\zeta_i}(W_i(x))) \cdot \zeta_p(i,j,g,\beta_i(x))\Big) \cdot \mathsf{ap}_{\zeta_j}(\beta_p(i,j,g,x)) \cdot \mathsf{ap}_{\zeta_j}(W_j(F_{i,j,g}(x)))^{-1} \\
&= \\
&\mathsf{ap}_{H_{i,j,g}}(\mathsf{ap}_{\zeta_i}(W_i(x))) \cdot \Big(\zeta_p(i,j,g,\beta_i(x)) \cdot \mathsf{ap}_{\zeta_j}(\beta_p(i,j,g,x))\Big) \cdot \mathsf{ap}_{\zeta_j}(W_j(F_{i,j,g}(x)))^{-1}
\end{aligned}$$

Proposition 7.0.5 (Concatenation). *Let $\alpha^*, \beta^*, \epsilon^* : F \Rightarrow G$. Let $(W,C) : \alpha^* \sim_\Gamma \beta^*$ and $(Y,D) : \beta^* \sim_\Gamma \epsilon^*$. We have a path*

$$\langle W, C\rangle \cdot \langle Y, D\rangle \;=_{\alpha^* = \epsilon^*}\; \langle \lambda i \lambda x. W_i(x) \cdot Y_i(x), \lambda i \lambda j \lambda g \lambda x. \tau_{C,D}(i,j,g,x)\rangle$$

where $\tau_{C,D}(i,j,g,x)$ denotes the chain of equalities

$$\begin{aligned}
&\alpha_p(i,j,g,x) \\
&= C_{i,j,g}(x) \\
&\mathsf{ap}_{G_{i,j,g}}(W_i(x)) \cdot \beta_p(i,j,g,x) \cdot W_j(F_{i,j,g}(x))^{-1} \\
&= \text{ via } D_{i,j,g}(x) \\
&\mathsf{ap}_{G_{i,j,g}}(W_i(x)) \cdot \Big(\mathsf{ap}_{G_{i,j,g}}(Y_i(x)) \cdot \epsilon_p(i,j,g,x) \cdot Y_j(F_{i,j,g}(x))^{-1}\Big) \cdot W_j(F_{i,j,g}(x))^{-1} \\
&= \\
&\mathsf{ap}_{G_{i,j,g}}(W_i(x) \cdot Y_i(x)) \cdot \epsilon_p(i,j,g,x) \cdot (W_j(F_{i,j,g}(x)) \cdot Y_j(F_{i,j,g}(x)))^{-1}
\end{aligned}$$

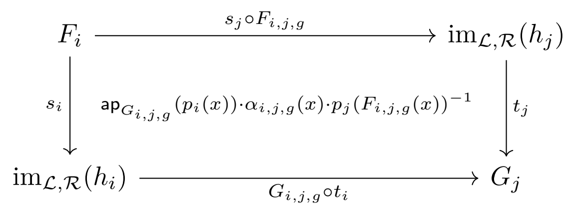
Lemma 7.0.6. *The category $\mathcal{D}_\Gamma$ is a bicategory.*

Proof. The associativity and unit laws of maps hold definitionally in $\mathcal{U}$. Thus, by Propositions 7.0.3 to 7.0.5 for $\mathcal{D}_\Gamma$ together with Lemma 7.0.2, verifying that $\mathcal{D}_\Gamma$ is a bicategory reduces to routine path algebra, which we omit here. □

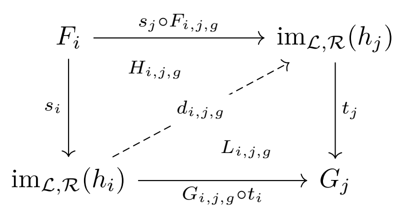
Proposition 7.0.7. *Assuming the univalence axiom, the category $\mathcal{D}_\Gamma$ is a univalent bicategory.*

Preservation theorem

Let F and G be $\mathcal{U}$-valued diagrams over Γ and suppose that $(h, \alpha) : F \Rightarrow G$. For each $i : \Gamma_0$, we have a factorization $(\mathrm{im}_{\mathcal{L},\mathcal{R}}(h_i), s_i, t_i, p_i, \ldots)$ of h_i. Therefore, we have a commuting square



and thus a diagonal filler



where

$$\mathsf{ap}_{G_{i,j,g}}(p_i(x)) \cdot \alpha_{i,j,g}(x) \cdot p_j(F_{i,j,g}(x))^{-1} \;=\; L_{i,j,g}(s_i(x))^{-1} \cdot \mathsf{ap}_{t_j}(H_{i,j,g}(x))$$

for all $x : F_i$.

Theorem 7.0.8 (Unique factorization). *For each $F, G : \mathcal{D}_\Gamma$ and $H : F \Rightarrow G$, define the predicates*

$$\widehat{\mathcal{L}}(H) \;:\equiv\; \prod_{i:\Gamma_0} \mathcal{L}(H_i)$$
$$\widehat{\mathcal{R}}(H) \;:\equiv\; \prod_{i:\Gamma_0} \mathcal{R}(H_i).$$

Let $(h, \alpha) : F \Rightarrow G$. The type

$$\mathsf{fact}_{\widehat{\mathcal{L}},\widehat{\mathcal{R}}}(h, \alpha) \;:\equiv\; \sum_{A:\mathcal{D}_\Gamma} \sum_{S:F\Rightarrow A} \sum_{T:A\Rightarrow G} (T \circ S \sim (h, \alpha)) \times \widehat{\mathcal{L}}(S) \times \widehat{\mathcal{R}}(T).$$

is contractible.

Proof. We have already exhibited an element of $\mathsf{fact}_{\widehat{\mathcal{L}},\widehat{\mathcal{R}}}(h, \alpha)$. Thus, it remains to prove that it's a mere proposition.

To this end, note that $\mathsf{fact}_{\widehat{\mathcal{L}},\widehat{\mathcal{R}}}(h,\alpha)$ is equivalent to the type of data

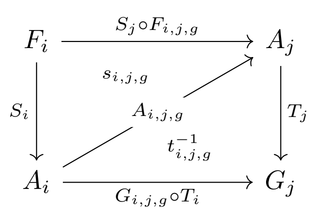
$$
\begin{aligned}
A_0 &: \Gamma_0 \to \mathcal{U} \\
S_0 &: \prod_{i:\Gamma_0} F_i \to A_i \\
T_0 &: \prod_{i:\Gamma_0} A_i \to G_i \\
P &: \prod_{i:\Gamma_0} T_i \circ S_i \sim h_i \\
L &: \prod_{i:\Gamma_0} \mathcal{L}(S_i) \\
R &: \prod_{i:\Gamma_0} \mathcal{R}(T_i)
\end{aligned}
\qquad
\begin{aligned}
A_1 &: \prod_{i,j:\Gamma_0} \prod_{g:\Gamma_1(i,j)} A_0(i) \to A_0(j) \\
S_1 &: \prod_{i,j,g} A_{i,j,g} \circ S_i \sim S_j \circ F_{i,j,g} \\
T_1 &: \prod_{i,j,g} G_{i,j,g} \circ T_i \sim T_j \circ A_{i,j,g} \\
p &: \prod_{i,j,g} \prod_{x:F_i} (T_1 * S_1)\,(i,j,g,x) = \mathsf{ap}_{G_{i,j,g}}(P_i(x)) \cdot \alpha_{i,j,g}(x) \cdot P_j(F_{i,j,g}(x))^{-1}
\end{aligned}
$$

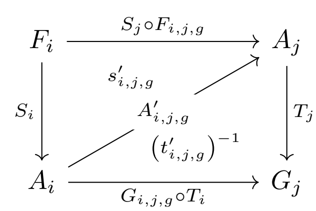
We can contract the six left-hand fields because $\mathsf{fact}_{\mathcal{L},\mathcal{R}}(h_i)$ is contractible for each $i : \Gamma_0$. Let (A_0, S, T, P, L, R) be a tuple of the first six fields and call the type of the last four fields $\mathsf{coher}_{\mathcal{L},\mathcal{R}}(A_0, S, T, P, L, R)$. Let (A, s, t, p) and (A', s', t', p') be elements of $\mathsf{coher}_{\mathcal{L},\mathcal{R}}(A_0, S, T, P, L, R)$. We must prove that they are equal.

Note that

$$
\begin{aligned}
p_{i,j,g}(x) &: t_{i,j,g}(S_i(x)) \cdot \mathsf{ap}_{T_j}(s_{i,j,g}(x)) = \mathsf{ap}_{G_{i,j,g}}(P_i(x)) \cdot \alpha_{i,j,g}(x) \cdot P_j(F_{i,j,g}(x))^{-1} \\
p'_{i,j,g}(x) &: t'_{i,j,g}(S_i(x)) \cdot \mathsf{ap}_{T_j}(s'_{i,j,g}(x)) = \mathsf{ap}_{G_{i,j,g}}(P_i(x)) \cdot \alpha_{i,j,g}(x) \cdot P_j(F_{i,j,g}(x))^{-1}
\end{aligned}
$$

Therefore, we have two commuting squares





along with two terms

$$
\left(A_{i,j,g}, s_{i,j,g}, t^{-1}_{i,j,g}, p_{i,j,g}\right),\ \left(A'_{i,j,g}, s'_{i,j,g}, \left(t'_{i,j,g}\right)^{-1}, p'_{i,j,g}\right) \ : \ \mathsf{fill}(\lambda x.\, \mathsf{ap}_{G_{i,j,g}}(P_i(x)) \cdot \alpha_{i,j,g}(x) \cdot P_j(F_{i,j,g}(x))^{-1})
$$

By Lemma 3.3.5 combined with Theorem A.0.3, this gives us data

$$
\begin{aligned}
\Delta^1_{i,j,g} &: A'_{i,j,g} \sim A_{i,j,g} \\
\Delta^2_{i,j,g}(x) &: \Delta^1_{i,j,g}(S_i(x))^{-1} \cdot s'_{i,j,g}(x) \ = \ s_{i,j,g}(x) \\
\Delta^3_{i,j,g}(x) &: t'_{i,j,g}(x) \cdot \mathsf{ap}_{T_j}(\Delta^1_{i,j,g}(x)) \ = \ t_{i,j,g}(x) \\
\Delta^4_{i,j,g}(x) &: \tau_{i,j,g}(x) \cdot \mathsf{ap}_{\left(t'_{i,j,g}(S_i(x)) \cdot \mathsf{ap}_{T_j}(\Delta^1_{i,j,g}(S_i(x)))\right) \cdot -}(\mathsf{ap}_{\mathsf{ap}_{T_j}}(\Delta^2_{i,j,g}(x))) \cdot \mathsf{ap}_{- \cdot \mathsf{ap}_{T_j}(s_{i,j,g}(x))}(\Delta^3_{i,j,g}(S_i(x))) \ = \ p'_{i,j,g}(x) \cdot p_{i,j,g}(x)^{-1}
\end{aligned}
$$

where $\tau_{i,j,g}(x)$ is the evident term of type

$$t'_{i,j,g}(S_i(x)) \cdot \mathsf{ap}_{T_j}(s'_{i,j,g}(x)) \;=\; \Big(t'_{i,j,g}(S_i(x)) \cdot \mathsf{ap}_{T_j}(\Delta^1_{i,j,g}(S_i(x)))\Big) \cdot \mathsf{ap}_{T_j}(\Delta^1_{i,j,g}(S_i(x))^{-1} \cdot s'_{i,j,g}(x))$$

By Theorem A.0.3 again, the type of such data is equivalent to $(A_1, s, t, p) = (A'_1, s', t', p')$, thereby completing the proof. □

Corollary 7.0.9. *The OFS $(\mathcal{L}, \mathcal{R})$ on $\mathcal{U}$ lifts pointwise to $\mathcal{D}_\Gamma$.*

Since the functor $\mathsf{const}_\Gamma : \mathcal{U} \to \mathcal{D}_\Gamma$ clearly takes $\mathcal{R}$ to $\widehat{\mathcal{R}}$, we deduce that $\mathsf{colim}_\Gamma(-)$ takes $\widehat{\mathcal{L}}$ to $\mathcal{L}$ by Corollary 3.3.9. Now, for each $X, Y : A/\mathcal{U}$, consider the predicate $\mathcal{L}_A(f, p) :\equiv \mathcal{L}(f)$ on $X \to_A Y$. Then the functor colim^A_Γ takes $\widehat{\mathcal{L}}_A$ to $\mathcal{L}_A$. Indeed, if a map $\epsilon : \mathcal{A} \Rightarrow \mathcal{B}$ of A-diagrams belongs to $\widehat{\mathcal{L}}_A$, then the underlying function of $\mathsf{colim}^A_\Gamma(\epsilon)$ is precisely that induced by the morphism of spans

$$\begin{array}{ccccc} A & \longleftarrow & \mathsf{colim}_\Gamma A & \longrightarrow & \mathsf{colim}_\Gamma(\mathcal{F}(\mathcal{A})) \\ \downarrow{\scriptstyle \mathsf{id}} & & \downarrow{\scriptstyle \mathsf{id}} & & \downarrow{\scriptstyle \hat{\epsilon}} \\ A & \longleftarrow & \mathsf{colim}_\Gamma A & \longrightarrow & \mathsf{colim}_\Gamma(\mathcal{F}(\mathcal{B})) \end{array}$$

and all three components belong to $\mathcal{L}$.

In particular, if F is a *pointed* diagram over Γ such that each $\mathsf{pr}_1(F_i)$ is $(\mathcal{L}, \mathcal{R})$-connected, then the type $\mathsf{colim}^*_\Gamma F$ is also $(\mathcal{L}, \mathcal{R})$-connected. Indeed, since $\mathsf{colim}^*_\Gamma \mathbf{1} = \mathbf{1}$, colim^*_Γ takes the unique map $F \Rightarrow_* \mathbf{1}$ of pointed diagrams to $(c, c_p) : \mathsf{colim}^*_\Gamma F \to_* \mathsf{colim}^*_\Gamma \mathbf{1}$ where $c : \mathsf{colim}^*_\Gamma F \to \mathbf{1}$ is the constant map and $c \in \mathcal{L}$.

Example 7.0.10.

(a) For each truncation level n, if each $\mathsf{pr}_1(F_i)$ is n-connected, then so is the underlying type of $\mathsf{colim}^*_\Gamma F$. In fact, if F is an A-diagram with each $\mathsf{pr}_1(F_i)$ n-connected and A is n-connected, then Corollary 5.5.3 shows that the underlying type of $\mathsf{colim}^A_\Gamma F$ is also n-connected.

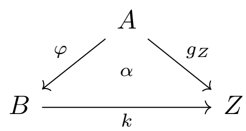
(b) Let Γ be the graph with a single point $*$ and a single edge from $*$ to itself. Define the diagram F over Γ by $F(*) :\equiv \mathbf{1}$ and $F_{*,*,*} :\equiv \mathsf{id}_{\mathbf{1}}$. Then $\mathsf{colim}_\Gamma(F) = S^1$, which proves that colim_Γ does not preserve n-connectedness when $n \geq 1$, unlike colim^*_Γ.

7.1 Colimits of higher groups

Let $-2 \leq n \leq \infty$ and $-1 \leq k < \infty$ be truncation levels. Recall from [5] the category $(n, k)\,\mathsf{GType}$ of *k-tuply groupal n-groupoids*, whose objects are known as *higher groups*. This category is isomorphic to the full subcategory $\mathcal{U}^*_{\geq k, \leq n+k}$ of $\mathcal{U}^*$ on $(k-1)$-connected, $(n+k)$-truncated types. For each truncation level m, consider the full subcategory of $A/\mathcal{U}$ on those objects whose underlying types are m-truncated. By Corollary 3.4.7, this subcategory inherits colimits from $A/\mathcal{U}$. We now conclude from Example 7.0.10(a) that $(n, k)\,\mathsf{GType}$ has colimits over graphs.

Remark. By our second construction of colim_Γ^A (Corollary 5.5.3), we have that for every OFS $(\mathcal{L}, \mathcal{R})$ on $\mathcal{U}$ and all $A : \mathcal{U}$, if A is $(\mathcal{L}, \mathcal{R})$-connected, then the full subcategory of $A/\mathcal{U}$ on $(\mathcal{L}, \mathcal{R})$-connected types has colimits over graphs.[8]

Let $Q : \mathcal{U} \to \mathsf{Prop}$. Let $(A, \_), (B, \_) : \mathcal{U}_Q$ and $\varphi : A \to B$. Consider the *coslice-coslice* (wild) category $(B, \varphi) / (A/\mathcal{U}_Q)$. Its objects are commuting triangles $((Z, \_), g_Z, k, \alpha)$



and its morphisms $(Z_1, g_{Z_1}, k_1, \alpha_1) \to_\varphi (Z_2, g_{Z_2}, k_2, \alpha_2)$ are tuples

$$
\begin{aligned}
f &: Z_1 \to Z_2 \\
p &: f \circ g_{Z_1} \sim g_{Z_2} \\
H &: f \circ k_1 \sim k_2 \\
K &: \prod_{a:A} p(a) = \mathsf{ap}_f(\alpha_1(a))^{-1} \cdot H(\varphi(a)) \cdot \alpha_2(a)
\end{aligned}
$$

Composition $\circ$ of morphisms is defined by

$$
\begin{aligned}
\circ \; : \; & ((Z_2, g_{Z_2}, k_2, \alpha_2) \to_\varphi (Z_3, g_{Z_3}, k_3, \alpha_3)) \to ((Z_1, g_{Z_1}, k_1, \alpha_1) \to_\varphi (Z_2, g_{Z_2}, k_2, \alpha_2)) \to \cdots \\
(f_2, p_2, H_2, K_2) \circ (f_1, p_1, H_1, K_1) \; &:\equiv \; \big(f_2 \circ f_1, \lambda a.\, \mathsf{ap}_{f_2}(p_1(a)) \cdot p_2(a), \lambda b.\, \mathsf{ap}_{f_2}(H_1(b)) \cdot H_2(b), \sigma(K_2, K_1)\big)
\end{aligned}
$$

where $\sigma(K_2, K_1, a)$ denotes the chain of equalities

$$
\begin{gathered}
\mathsf{ap}_{f_2}(p_1(a)) \cdot p_2(a) \\
\big\| \; \text{via } K_1(a) \\
\mathsf{ap}_{f_2}(\mathsf{ap}_{f_1}(\alpha_1(a))^{-1} \cdot H_1(\varphi(a)) \cdot \alpha_2(a)) \cdot p_2(a) \\
\big\| \; \text{via } K_2(a) \\
\mathsf{ap}_{f_2}(\mathsf{ap}_{f_1}(\alpha_1(a))^{-1} \cdot H_1(\varphi(a)) \cdot \alpha_2(a)) \cdot \big(\mathsf{ap}_{f_2}(\alpha_2(a))^{-1} \cdot H_2(\varphi(a)) \cdot \alpha_3(a)\big) \\
\big\| \\
\mathsf{ap}_{f_2 \circ f_1}(\alpha_1(a))^{-1} \cdot \big(\mathsf{ap}_{f_2}(H_1(\varphi(a))) \cdot H_2(\varphi(a))\big) \cdot \alpha_3(a)
\end{gathered}
$$

for each $a : A$.

We now define 0-functors

$$
(B, \varphi) / (A/\mathcal{U}_Q) \underset{\xi}{\overset{\gamma}{\rightleftarrows}} B/\mathcal{U}_Q
$$

[8] If $(\mathcal{L}, \mathcal{R})$ is an OFS on $\mathcal{U}$, then a type is $(\mathcal{L}, \mathcal{R})$-*connected* if it is such for the underlying reflective subcategory of $(\mathcal{L}, \mathcal{R})$ (see [11, Lemma 2.32]).

Define $\gamma_0 : \mathsf{Ob}((B,\varphi)\,/\,(A/\mathcal{U}_Q)) \to \mathsf{Ob}(B/\mathcal{U}_Q)$ by $\gamma_0(Z, g_Z, k, \alpha) :\equiv (Z, k)$. Conversely, define $\xi_0 : \mathsf{Ob}(B/\mathcal{U}_Q) \to \mathsf{Ob}((B,\varphi)\,/\,(A/\mathcal{U}_Q))$ by $\xi_0(Z,k) :\equiv \big(Z, k\circ\varphi, k, \lambda a.\mathsf{refl}_{k(\varphi(a))}\big)$. Next, define the functions

$$\begin{aligned}
\gamma_1 &: \mathsf{hom}_{(B,\varphi)/(A/\mathcal{U}_Q)}((Z_1, g_{Z_1}, k_1, \alpha_1), (Z_2, g_{Z_2}, k_2, \alpha_2)) \to \mathsf{hom}_{B/\mathcal{U}_Q}((Z_1,k_1),(Z_2,k_2)) \\
\gamma_1(f,p,H,k) &:\equiv (f, H) \\
\xi_1 &: \mathsf{hom}_{B/\mathcal{U}_Q}((Z_1,k_1),(Z_2,k_2)) \to \mathsf{hom}_{(B,\varphi)/(A/\mathcal{U}_Q)}\big(\big(Z_1, k_1\circ\varphi, k_1, \lambda a.\mathsf{refl}_{k_1(\varphi(a))}\big), \big(Z_2, k_2\circ\varphi, k_2, \lambda a.\mathsf{refl}_{k_2(\varphi(a))}\big)\big) \\
\xi_1(f,H) &:\equiv \big(f, \lambda a.H(\varphi(a)), H, \lambda a.\,\mathsf{RId}(H(\varphi(a)))^{-1}\big)
\end{aligned}$$

Lemma 7.1.1. *We have that $\xi \dashv \gamma$.*

Proof. Let $(Z_1, g_{Z_1}, k_1, \alpha) : \mathsf{Ob}((B,\varphi)\,/\,(A/\mathcal{U}_Q))$ and $(Z_2,k_2) : \mathrm{Ob}(B/\mathcal{U}_Q)$. Define

$$\begin{aligned}
\mu &: \mathsf{hom}_{(B,\varphi)/(A/\mathcal{U}_Q)}\big(\big(Z_2, k_2\circ\varphi, k_2, \lambda a.\mathsf{refl}_{k_2(\varphi(a))}\big), (Z_1, g_{Z_1}, k_1, \alpha)\big) \to \mathsf{hom}_{B/\mathcal{U}_Q}((Z_2,k_2),(Z_1,k_1)) \\
\mu(f,p,H,K) &:\equiv (f,H)
\end{aligned}$$

We claim that μ is contractible, hence an equivlanece. Indeed, for each $(g, I) : \mathsf{hom}_{B/\mathcal{U}_Q}((Z_2,k_2),(Z_1,k_1))$,

$$\begin{aligned}
&\mathsf{fib}_\mu(g,I) \\
\simeq\ & \sum_{f:Z_2\to Z_1}\ \sum_{p:f\circ k_2\circ\varphi\sim g_{Z_1}}\ \sum_{H:f\circ k_2\sim k_1}\ \sum_{K:\prod_{a:A} p(a)=H(\varphi(a))\cdot\alpha(a)}\ \sum_{U:f\sim g}\ \prod_{b:B} U(k_2(b))^{-1}\cdot H(b) = I(b) \\
\simeq\ & \mathbf{1} \qquad \textit{(strong function extensionality)}
\end{aligned}$$

Moreover, it is easy to check that μ is natural in both variables and that the resulting adjunction is 2-coherent. □

Lemma 7.1.2. *The category $(B,\varphi)\,/\,(A/\mathcal{U}_Q)$ is reflective in $B/\mathcal{U}_Q$, i.e., it admits a 2-coherent left adjoint from $B/\mathcal{U}_Q$ whose counit is a natural isomorphism.*

Proof. The counit $\epsilon : \xi\circ\gamma \to \mathsf{id}_{(B,\varphi)/(A/\mathcal{U}_Q)}$ of the adjunction of Lemma 7.1.1 is an equivalence in each component:

$$\begin{aligned}
\epsilon_{Z^*} &: \big(Z, k\circ\varphi, k, \lambda a.\mathsf{refl}_{k(\varphi(a))}\big) \xrightarrow{\simeq}_{\varphi} (Z, g_Z, k, \alpha) \\
\epsilon_{Z^*} &:\equiv \big(\mathsf{id}_Z, \alpha, \lambda b.\mathsf{refl}_{k(b)}, \lambda a.\mathsf{refl}_{\alpha(a)}\big)
\end{aligned}$$

As ϵ is automatically a natural transformation of 0-functors, it follows that it is a natural isomorphism. □

Corollary 7.1.3. *Consider truncation levels $-2\le n\le\infty$ and $-1\le k<\infty$. For each object G of $(n,k)\,\mathsf{GType}$, the coslice $G/(n,k)\,\mathsf{GType}$ is cocomplete on graphs.*

Proof. The category $G/\mathcal{U}^*_{\ge k,\le n+k}$ is reflective in $\mathsf{pr}_1(G)/\mathcal{U}_{\ge k,\le n+k}$, which has colimits over graphs. □

Let $n, m : \mathbb{N}$ with $n > 0$ and $m < n$. The Eilenberg-MacLane space $K(\mathbb{Z}, n)$ is the free group on one generator in the category $(n, m)\,\mathsf{GType}$. Therefore, when $m > 0$, we may view the coslice-coslice $K(\mathbb{Z}, n)/\mathcal{U}^*_{\geq m, \leq n+m}$ as a higher version of the category of *pointed abelian groups* [9]. By Corollary 7.1.3, this category is cocomplete over graphs.

8 Weak continuity of cohomology

For this section, we need the notion of *finite graph.*

Definition 8.0.1 (Finite graph).

- We say that a type A is *finite* if it's merely equivalent to a standard finite type.
- We say that a graph Γ is *finite* if Γ_0 is finite and for each $i, j : \Gamma_0$, $\Gamma_1(i, j)$ is finite.

Lemma 8.0.2. *If Γ is a finite graph, then the type $\sum_{i,j:\Gamma_0} \Gamma_1(i, j)$ is finite.*

Proof. This follows directly from [12, A dependent sum of finite types indexed by a finite type is finite]. ☐

Let Γ be a finite graph. We claim that every generalized cohomology theory $k : (\mathcal{U}^*)^{\mathrm{op}} \to \mathbf{Ab}$ takes pointed colimits over Γ to weak limits in **Set**, in the sense that the universal map from the limit is epic in **Ab** (i.e., surjective). If k is additive (e.g., induced by an Ω-spectrum), then this holds when Γ is a 0-*choice* graph, i.e., Γ_0 and $\Gamma_1(i, j)$ satisfy the set-level axiom of choice [4, Definition 6.1].

8.1 Eilenberg-Steenrod cohomology

Let H be a $\mathbb{Z}$-indexed familiity of functors $(\mathcal{U}^*)^{\mathrm{op}} \to \mathbf{Ab}$. We say that H is an *(Eilenberg-Steenrod) cohomology theory* if it satisfies the following two axioms.

- For each $n : \mathbb{Z}$, we have a natural isomorphism $H^{n+1}(\Sigma -) \xrightarrow{\sigma_n} H^n(-)$ of functors $\mathcal{U}^* \to \mathbf{Ab}$.
- For each $f : X \to_* Y$, the following sequence is exact:

$$H^n({}^{Y}\!/_{X}) \xrightarrow{H^n(\mathsf{cof}(f))} H^n(Y) \xrightarrow{H^n(f)} H^n(X)$$

Example 8.1.1. Suppose that $E : \mathbb{Z} \to \mathcal{U}^*$ is a prespectrum, with structure maps $\epsilon_n : E_n \to_* \Omega E_{n+1}$. For each $n : \mathbb{Z}$, we have a sequence

$$\left\| X \to_* \Omega^k E_{n+k} \right\|_0 \xrightarrow{\left\| \Omega^k(\epsilon_{n+k}) \circ - \right\|_0} \left\| X \to_* \Omega^{k+1} E_{n+(k+1)} \right\|_0$$

of abelian groups.[9] For each $n : \mathbb{Z}$, define

$$\begin{aligned} \widetilde{E}^n &: \mathcal{U}^* \to \mathbf{Ab} \\ \widetilde{E}^n(X) &:= \mathsf{colim}_{k:\omega} \left\| X \to_* \Omega^k E_{n+k} \right\|_0 \end{aligned}$$

where the colimit $\mathsf{colim}_{k:\omega}(G_k)$ of a sequence of abelian groups has underlying set $\mathsf{colim}_k(\mathsf{pr}_1(G_k))$ and has abelian group structure defined by induction on sequentual colimits. This is a cohomology theory. The suspension axiom is easy to verify. We now turn to verifying the exactness axiom.

Definition 8.1.2. Let (A, a) be a $\mathcal{U}$-valued sequential digram. Let $n : \mathbb{N}$ and $x : A_n$. Define the *lifting* function

$$\begin{aligned} x^{(-)} &: (m : \mathbb{N}) \to A_{n+m} \\ x^0 &:= x \\ x^{m+1} &:= a_{n+m}(x^m) \end{aligned}$$

where $+ : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$ is defined by pattern matching on the second argument.

Lemma 8.1.3. *Consider a levelwise exact sequence*

$$(A, a) \xrightarrow{(m_1, M_1)} (B, b) \xrightarrow{(m_2, M_2)} (C, c)$$

of sequential diagrams valued in $\mathbf{Ab}$*. Then the following sequence of abelian groups is exact:*

$$\mathsf{colim}_{k:\omega}(A_k) \xrightarrow{\mathsf{colim}(m_1)} \mathsf{colim}_{k:\omega}(B_k) \xrightarrow{\mathsf{colim}(m_2)} \mathsf{colim}_{k:\omega}(C_k)$$

Proof. For each $k : \mathbb{N}$ and $x : A_k$,

$$\mathsf{colim}(m_2 \circ m_1)(\iota_k(x)) \equiv \iota_k(m_2(k, (m_1(k, x)))) = 0$$

because $m_2(k) \circ m_1(k) = 0$ by levelwise exactness.

Next, let $k : \mathbb{N}$ and $x : B_k$. Suppose that $\overbrace{\mathsf{colim}(m_2)(\iota_k(x))}^{\iota_k(m_2(k,x))} = 0$. We want to show that the fiber of $\mathsf{colim}(m_1)$ over $\iota_k(x)$ is nonempty. By [13, Theorem 7.4], we have an equivalence

$$(\iota_k(0) =_{\mathsf{colim}\, C} \iota_k(m_2(k, x))) \simeq \mathsf{colim}_{n:\omega}(0^{+n} =_{C_{k+n}} m_2(k, x)^{+n})$$

As $\iota_k(0) = 0$, it thus suffices to prove that the fiber is nonempty given a term of type $\mathsf{colim}_n(0^{+n} = m_2(k, x)^{+n})$. We proceed by induction on sequential colimits. Let $n : \mathbb{N}$ and $p : 0^{+n} = m_2(k, x)^{+n}$. By naturality of m_2, we see that $m_2(k, x)^{+n} = m_2(k + n, x^{+n})$. As $0^{+n} = 0$, it follows that x^{+n} belongs to the kernel of $m_2(k + n)$. By levelwise exactness, this gives us an element $(d, q) : \mathsf{fib}_{m_1(k+n)}(x^{+n})$.

[9] We assume that addition $+ : \mathbb{Z} \times \mathbb{N} \to \mathbb{Z}$ is defined by pattern matching on the second argument.

We have that

$$\mathsf{colim}(m_1)(\iota_{k+n}(d)) \;\equiv\; \iota_{k+n}(m_1(k+n,d)) \;=\; \iota_{k+n}(x^{+n}) \;=\; \iota_k(x).$$

This proves that the fiber over $\iota_k(x)$ is nonempty. ☐

Now it suffices to observe that

$$\left\|{}^{Y}\!/_{X} \to_* \Omega^k E_{n+k}\right\|_0 \xrightarrow{\|-\circ \mathsf{cof}(f)\|_0} \left\|Y \to_* \Omega^k E_{n+k}\right\|_0 \xrightarrow{\|-\circ f\|_0} \left\|X \to_* \Omega^k E_{n+k}\right\|_0$$

is exact for every $f : X \to_* Y$ (see [6, Section 3.2.2]).

For each $n : \mathbb{Z}$, the functor $\widetilde{E}^n(-)$ computes the -$2n$-th degree $[\Sigma^\infty(-), E]_{-2n}$ of the graded hom-group in the category of prespectra [1, Proposition 2.8], where $\Sigma^\infty(X)$ denotes the suspension prespectrum of a pointed type X. For example, if E is the sphere spectrum, then $\widetilde{E}^{-n}(-)$ is precisely the $2n$-th homotopy group functor $\pi_{2n}(-)$ on prespectra.

If H satisfies

$$H^n(S^0) \cong \mathbf{1}$$

for all $n \neq 0$, then H is called *ordinary.*

If the map

$$\prod_{i:I} H^n(\mathsf{inr} \circ (i,-)) \;:\; H^n(\bigvee_{i:I} F_i) \to \prod_{i:I} H^n(F_i)$$

is an isomorphism for every set I satisfying the set-level axiom of choice and every family $F : I \to \mathcal{U}^*$ of pointed types, then H is called *additive.*

Example 8.1.4. Every Ω-spectrum $E : \mathbb{Z} \to \mathcal{U}^*$ induces an additive cohomology theory $\widetilde{E}$, which is ordinary if E is an Eilenberg-MacLane spectrum.

8.2 Cohomology sends finite colimits to weak limits

Suppose that H^* is a cohomology theory. Consider a pushout

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & \longrightarrow & P \end{array}$$

of pointed types and pointed maps. In [6], Cavallo constructs a LES of the form

$$\cdots \longrightarrow H^{n-1}(Z) \xrightarrow{\mathsf{extglue}} H^n(P) \xrightarrow{(H^n(\mathsf{inl}),H^n(\mathsf{inr}))} H^n(X) \times H^n(Y) \xrightarrow{H^n(f)-H^n(g)} H^n(Z) \longrightarrow \cdots$$

In particular, thanks to Corollary 5.5.3 combined with Lemma 8.0.2, if Γ is a finite graph, then we have an exact squence

$$H^n(\mathsf{colim}^*_\Gamma F) \xrightarrow{\zeta_n} \prod_{i,j,g} H^n(F_i) \times \prod_i H^n(F_i) \xrightarrow{\mu_n - \nu_n} \prod_{i,j,g} H^n(F_i) \times \prod_{i,j,g} H^n(F_i) \qquad (\texttt{ES})$$

for each $n : \mathbb{N}$. If H^* is additive, then this holds when Γ is just a 0-choice graph. Here, ζ_n is defined as the composite

$$\begin{array}{ccc} H^n(\mathsf{colim}^*_\Gamma F) & \overset{\zeta_n}{\dashrightarrow} & \prod_{i,j,g} H^n(F_i) \times \prod_i H^n(F_i) \\ {\scriptstyle (H^n(\mathsf{inl}), H^n(\mathsf{inr}))} \big\downarrow & \nearrow {\scriptstyle \cong \times \cong} & \\ H^n(\bigvee_{i,j,g} F_i) \times H^n(\bigvee_i F_i) & & \end{array}$$

and μ_n and ν_n are defined as

$$\begin{aligned} (f, h) &\mapsto (f, \lambda i \lambda j \lambda g. H^n(F_{i,j,g})(h_j)) \\ (f, h) &\mapsto (f, \lambda i \lambda j \lambda g. h_i) \end{aligned}$$

respectively. We have a cone $\mathcal{M}_{F,H,n}$

$$\begin{array}{ccccc} & & H^n(\mathsf{colim}^*_\Gamma(F)) & & \\ & \swarrow {\scriptstyle H^n(\iota_j)} & & {\scriptstyle H^n(\iota_i)} \searrow & \\ H^n(F_j) & & \xrightarrow[H^n(F_{i,j,g})]{} & & H^n(F_i) \end{array}$$

over $H^n(F)$ and thus a commuting diagram

$$\begin{array}{ccccc} H^n(\mathsf{colim}^*_\Gamma(F)) & & \overset{\Delta_F}{\dashrightarrow} & & \mathsf{lim}_\Gamma H^n(F) \\ & {\scriptstyle H^n(\iota_i)} \searrow & & \swarrow {\scriptstyle \mathsf{pr}_i} & \\ & & H^n(F_i) & & \end{array}$$

induced by the universal property of limits in **Ab**. Thanks to the exact sequence ($\texttt{ES}$), we see that Δ_F is epic. As a result, Δ_F is epic in **Set** as well. Classically, this implies that Δ_F has a section, so that $H^n(\mathsf{colim}^*_\Gamma(F))$ is a *weak limit* of $H^n(F)$ in **Set**. If we assume the axiom of choice inside HoTT, then Δ_F *merely* has a section. In this case, we can conclude that $H^n(\mathsf{colim}^*_\Gamma(F))$ is merely a weak limit in **Set**, i.e., the function

$$(\mathcal{M}_{F,H,n} \circ -) \; : \; (X \to H^n(\mathsf{colim}^*_\Gamma(F))) \to \mathsf{Cone}_{H^n(F)}(X)$$

is surjective (not nescessarily split) for each set X.

A Structure identity principle

We present our main tool for characterizing path spaces of structured types.

Definition A.0.1 (Identity system). Let (A, a) be a pointed type. Consider a type family B over A and an element $b : B(a)$. We say that (B, b) is an *identity system* on (A, a) if the total space $\sum_{x:A} B(x)$ is contractible.

Theorem A.0.2 ([10, Theorem 11.2.2]). *The following are logically equivalent.*

- *The family B is an identity system on (A, a).*
- *The family $f : \prod_{x:A} (a = x) \to B(x)$ defined by $f(a, \mathsf{refl}_a) :\equiv b$ is a family of equivalences.*
- *For each family of types $P : \prod_{a:A} B(x) \to \mathcal{U}$, the function*

$$h \mapsto h(a, b) \ : \ \left(\prod_{x:A} \prod_{y:B(x)} P(x, y) \right) \to P(a, b)$$

  has a section.

Theorem A.0.3 ([12, The structure identity principle]). *Let (A, a) be a pointed type, (B, b) a pointed type family over A, and (C, c) an identity system on (A, a). Consider terms*

$$D \ : \ \prod_{x:A} B(x) \to C(x) \to \mathcal{U} \qquad d \ : \ D(a, b, c)$$

If $\sum_{y:B(a)} D(a, y, c)$ is contractible, then the type family

$$(x, y) \ \mapsto \ \sum_{z:C(x)} D(x, y, z)$$

is an identity system on $(\sum_{x:A} B(x), (a, b))$.

B Left adjoints preserve colimits

We prove that 2-*coherent* left adjoints between wild categories preserve colimits over graphs.

Definition B.0.1. Suppose that $L : \mathcal{C} \to \mathcal{D}$ is a functor of wild categories and that $(\alpha, V_1, V_2) : L \dashv R$ (see Definition 3.1.7). We say that L is *2-coherent* if for all $h_1 : \mathsf{hom}_{\mathcal{D}}(L(X), Y)$, $h_2 : \mathsf{hom}_{\mathcal{C}}(Z, X)$, and $h_3 : \mathsf{hom}_{\mathcal{C}}(W, Z)$, the following diagram commutes:

$$
\begin{array}{ccc}
(\alpha(h_1) \circ h_2) \circ h_3 & \overset{\mathsf{assoc}(\alpha(h_1),h_2,h_3)}{=\!=\!=\!=\!=\!=} & \alpha(h_1) \circ (h_2 \circ h_3) \\
{\scriptstyle \mathsf{ap}_{-\circ h_3}(V_2(h_2,h_1))} \Big\| & & \Big\| {\scriptstyle V_2(h_2 \circ h_3, h_1)} \\
\alpha(h_1 \circ L(h_2)) \circ h_3 & & \alpha(h_1 \circ L(h_2 \circ h_3)) \\
{\scriptstyle V_2(h_3, h_1 \circ L(h_2))} \Big\| & & \Big\| {\scriptstyle \mathsf{ap}_{\alpha}(\mathsf{ap}_{h_1 \circ -}(L_2(h_2,h_3)))} \\
\alpha((h_1 \circ L(h_2)) \circ L(h_3)) & \underset{\mathsf{ap}_{\alpha}(\mathsf{assoc}(h_1, L(h_2), L(h_3)))}{=\!=\!=\!=\!=\!=} & \alpha(h_1 \circ (L(h_2) \circ L(h_3)))
\end{array}
$$

Let $\mathcal{C}$ be a wild category and Γ be a graph. Let F be a Γ-shaped diagram in $\mathcal{C}$. We say that a cocone (C, r, K) under F is *colimiting* if for all $X : \mathsf{Ob}(\mathcal{C})$,

$$
\begin{aligned}
\mathsf{postcomp}(C, r, K, X) &: \mathsf{hom}_{\mathcal{C}}(C, X) \to \mathsf{lim}_{i:\Gamma^{\mathrm{op}}}(\mathsf{hom}_{\mathcal{C}}(F_i, X)) \\
\mathsf{postcomp}(C, r, K, X, f) &:= \big(\lambda i. f \circ r_i, \lambda j \lambda i \lambda g.\, \mathsf{assoc}(f, r_j, F_{i,j,g}) \cdot \mathsf{ap}_{f \circ -}(K_{i,j,g})\big)
\end{aligned}
$$

is an equivalence. Let $\mathcal{D}$ be a wild category. Suppose that $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ are functors of wild categories and that $(\alpha, V_1, V_2) : L \dashv R$. Suppose that this adjunction is 2-coherent. Let (C, r, K) be a colimiting cocone under F. We have an induced cocone

$$
\begin{array}{ccc}
L_0(F_i) & \xrightarrow{\quad L_1(F_{i,j,g}) \quad} & L_0(F_j) \\
& {\scriptstyle L(K_{i,j,g})} & \\
{\scriptstyle L_1(r_i)} \searrow & & \swarrow {\scriptstyle L_1(r_j)} \\
& L_0(C) &
\end{array}
$$

under $L(F)$. Here, $L(K_{i,j,g}) :\equiv L_2(r_j, F_{i,j,g})^{-1} \cdot \mathsf{ap}_{L_1}(K_{i,j,g})$.

Theorem B.0.2. *The cocone $(L_0(C), L_1(r), L(K))$ under $L(F)$ is colimiting.*

Proof. For all $Y : \mathsf{Ob}(\mathcal{D})$, we have that

$$
\begin{aligned}
& \mathsf{hom}_{\mathcal{D}}(L_0(C), Y) \\
\simeq\ & \mathsf{hom}_{\mathcal{C}}(C, R_0(Y)) && (\alpha(C, Y)) \\
\simeq\ & \mathsf{lim}_{i:\Gamma^{\mathrm{op}}}(\mathsf{hom}_{\mathcal{C}}(F_i, R_0(Y))) && (\mathsf{postcomp}(C, r, K, R_0(Y))) \\
\simeq\ & \mathsf{lim}_{i:\Gamma^{\mathrm{op}}}(\mathsf{hom}_{\mathcal{D}}(L_0(F_i), Y)) && (\mathsf{lim}_{i:\Gamma^{\mathrm{op}}}(\alpha(F_i, Y)^{-1}))
\end{aligned}
$$

Let ζ denote the composite of these three functions. For each $h : \mathsf{hom}_{\mathcal{D}}(L_0(C), Y)$,

$$
\begin{array}{c}
\zeta(h) \\
\Big\| {\scriptstyle \textit{definitional}} \\
\Big(\lambda i. \alpha(F_i, Y)^{-1}(\alpha(C, Y, h) \circ r_i), \lambda j \lambda i \lambda g. \widetilde{V}_2(F_{i,j,g}, \alpha(C, Y, h) \circ r_j) \cdot \mathsf{ap}_{\alpha(F_i, Y)^{-1}}(\mathsf{assoc}(\alpha(C, Y, h), r_j, F_{i,j,g}) \cdot \mathsf{ap}_{\alpha(C,Y,h)\circ -}(K_{i,j,g}))\Big)
\end{array}
$$

For each $i : \Gamma_0$, we have a chain A_i of equalities

$$
\begin{aligned}
& \alpha(F_i, Y)^{-1}(\alpha(C, Y, h) \circ r_i) \\
&= \alpha^{-1}(C, Y, \alpha(C, Y, h)) \circ L(r_i) && (\mathsf{ap}_{\alpha(C,Y,h)^{-1}}(V_2(r_i, h))) \\
&= h \circ L(r_i) && (\eta_\alpha(h \circ L(r_i)))
\end{aligned}
$$

We want to prove that

$$
\begin{gathered}
\mathsf{ap}_{-\circ L_1(F_{i,j,g})}(A_j)^{-1} \cdot \widetilde{V}_2(F_{i,j,g}, \alpha(C, Y, h) \circ r_j) \cdot \mathsf{ap}_{\alpha(F_i,Y)^{-1}}(\mathsf{assoc}(\alpha(C, Y, h), r_j, F_{i,j,g}) \cdot \mathsf{ap}_{\alpha(C,Y,h)\circ -}(K_{i,j,g})) \cdot A_i \\
\Vert \\
\mathsf{assoc}(h, L_1(r_j), L_1(F_{i,j,g})) \cdot \mathsf{ap}_{h\circ -}(L(K_{i,j,g}))
\end{gathered}
$$

By Note 3.1.8, we have the commuting diagram

$$
\begin{array}{ccc}
\alpha^{-1}(\alpha(h) \circ r_j) \circ L(F_{i,j,g}) & \xRightarrow{\widetilde{V}_2(F_{i,j,g}, \alpha(h)\circ r_j)} & \alpha^{-1}((\alpha(h) \circ r_j) \circ F_{i,j,g}) \\
{\scriptstyle \mathsf{ap}_{\alpha^{-1}(-)\circ L(F_{i,j,g})}(V_2(r_j,h))} \Downarrow & \textit{homotopy naturality} & \Downarrow {\scriptstyle \mathsf{ap}_{\alpha^{-1}(-\circ F_{i,j,g})}(V_2(r_j,h))} \\
\alpha^{-1}(\alpha(h \circ L(r_j))) \circ L(F_{i,j,g}) & \xRightarrow[\widetilde{V}_2(F_{i,j,g}, \alpha(h\circ L(r_j)))]{} & \alpha^{-1}(\alpha(h \circ L(r_j)) \circ F_{i,j,g}) \\
{\scriptstyle \mathsf{ap}_{-\circ L(F_{i,j,g})}(\eta_\alpha(h\circ L(r_j)))} \Downarrow & \mathsf{exch}(F_{i,j,g}, h\circ L(r_j)) & \Downarrow {\scriptstyle \mathsf{ap}_{\alpha^{-1}}(V_2(F_{i,j,g}, h\circ L(r_j)))} \\
(h \circ L(r_j)) \circ L(F_{i,j,g}) & \xRightarrow[\eta_\alpha((h\circ L(r_j))\circ L(F_{i,j,g}))^{-1}]{} & \alpha^{-1}(\alpha((h \circ L(r_j)) \circ L(F_{i,j,g})))
\end{array}
$$

We also have the commuting diagram

$$
\begin{array}{ccc}
\alpha^{-1}(\alpha(h) \circ (r_j \circ F_{i,j,g})) & \xRightarrow{\mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_{\alpha(h)\circ -}(K_{i,j,g}))} & \alpha^{-1}(\alpha(h) \circ r_i) \\
{\scriptstyle \mathsf{ap}_{\alpha^{-1}}(V_2(r_j\circ F_{i,j,g}, h))} \Downarrow & \textit{homotopy naturality} & \Downarrow {\scriptstyle \mathsf{ap}_{\alpha^{-1}}(V_2(r_i,h))} \\
\alpha^{-1}(\alpha(h \circ L(r_j \circ F_{i,j,g}))) & \xRightarrow[\mathsf{ap}_{\alpha^{-1}}(\mathsf{ap}_\alpha(\mathsf{ap}_{h\circ L(-)}(K_{i,j,g})))]{} & \alpha^{-1}(\alpha(h \circ L(r_i))) \\
{\scriptstyle \eta_\alpha(h\circ L(r_j\circ F_{i,j,g}))} \Downarrow & \textit{homotopy naturality} & \Downarrow {\scriptstyle \eta_\alpha(h\circ L(r_i))} \\
h \circ L(r_j \circ F_{i,j,g}) & \xRightarrow[\mathsf{ap}_{h\circ -}(\mathsf{ap}_L(K_{i,j,g}))]{} & h \circ L(r_i)
\end{array}
$$

Thus, it suffices to prove that

$$
\begin{array}{ccc}
\alpha^{-1}((\alpha(h)\circ r_j)\circ F_{i,j,g}) & \overset{\mathsf{ap}_{\alpha^{-1}}(\mathsf{assoc}(\alpha(h),r_j,F_{i,j,g}))}{=\!=\!=\!=\!=} & \alpha^{-1}(\alpha(h)\circ(r_j\circ F_{i,j,g})) \\
\mathsf{ap}_{\alpha^{-1}(-\circ F_{i,j,g})}(V_2(r_j,h))\Big\Vert & & \Big\Vert \mathsf{ap}_{\alpha^{-1}}(V_2(r_j\circ F_{i,j,g},h)) \\
\alpha^{-1}(\alpha(h\circ L(r_j))\circ F_{i,j,g}) & & \alpha^{-1}(\alpha(h\circ L(r_j\circ F_{i,j,g}))) \\
\mathsf{ap}_{\alpha^{-1}}(V_2(F_{i,j,g},h\circ L(r_j)))\Big\Vert & & \Big\Vert \eta_\alpha(h\circ L(r_j\circ F_{i,j,g})) \\
\alpha^{-1}(\alpha((h\circ L(r_j))\circ L(F_{i,j,g}))) & & h\circ L(r_j\circ F_{i,j,g}) \\
\eta_\alpha((h\circ L(r_j))\circ L(F_{i,j,g}))\Big\Vert & & \Big\Vert \mathsf{ap}_{h\circ -}(L_2(r_j,F_{i,j,g})) \\
(h\circ L(r_j))\circ L(F_{i,j,g}) & \underset{\mathsf{assoc}(h,L(r_j),L(F_{i,j,g}))}{=\!=\!=\!=\!=} & h\circ(L(r_j)\circ L(F_{i,j,g}))
\end{array}
$$

commutes, i.e., the string of paths above the dashed line equals the string of paths below it. Now, the following diagram commutes by the 2-coherence of (α, V_1, V_2):

$$
\begin{array}{ccc}
(\alpha(h)\circ r_j)\circ F_{i,j,g} & \overset{\mathsf{assoc}(\alpha(h),r_j,F_{i,j,g})}{=\!=\!=\!=\!=} & \alpha(h)\circ(r_j\circ F_{i,j,g}) \\
\mathsf{ap}_{-\circ F_{i,j,g}}(V_2(r_j,h))\Big\Vert & & \Big\Vert V_2(r_j\circ F_{i,j,g},h) \\
\alpha(h\circ L(r_j))\circ F_{i,j,g} & & \alpha(h\circ L(r_j\circ F_{i,j,g})) \\
V_2(F_{i,j,g},h\circ L(r_j))\Big\Vert & & \Big\Vert \mathsf{ap}_\alpha(\mathsf{ap}_{h\circ -}(L_2(r_j,F_{i,j,g}))) \\
\alpha((h\circ L(r_j))\circ L(F_{i,j,g})) & \underset{\mathsf{ap}_\alpha(\mathsf{assoc}(h,L(r_j),L(F_{i,j,g})))}{=\!=\!=\!=\!=} & \alpha(h\circ(L(r_j)\circ L(F_{i,j,g})))
\end{array}
$$

The homotopy naturality of η_α now implies the desired equality. $\square$

Example B.0.3. The suspension-loop adjunction $\Sigma \dashv \Omega$ is defined component-wise by

$$
\begin{aligned}
\Phi \ &: \ \mathsf{hom}_{\mathcal{U}^*}(\Sigma(X,x_0),(Y,y_0)) \xrightarrow{\simeq} \mathsf{hom}_{\mathcal{U}^*}((X,x_0),\Omega(Y,y_0)) \\
\Phi(f,f_p) \ &:= \ (\lambda x.\underbrace{f_p^{-1}\cdot \mathsf{ap}_f(\mathsf{glue}_{\Sigma X}(x)\cdot \mathsf{glue}_{\Sigma X}(x_0)^{-1})\cdot f_p}_{\nu(x,f,f_p)}, \tau(\mathsf{glue}_{\Sigma X}(x_0),f_p))
\end{aligned}
$$

where $\tau(\mathsf{glue}_{\Sigma X}(x_0), f_p)$ denotes the evident term of type

$$f_p^{-1}\cdot \mathsf{ap}_f(\mathsf{glue}_{\Sigma X}(x_0)\cdot \mathsf{glue}_{\Sigma X}(x_0)^{-1})\cdot f_p = \mathsf{refl}_{y_0}$$

For every $(h,h_p) : \mathsf{hom}_{\mathcal{U}^*}((Z,z_0),(X,x_0))$, the naturality proof $V_2((h,h_p),(f,f_p))$ is the path

$$
\begin{aligned}
& \Phi(f,f_p)\circ(h,h_p) \\
\equiv\ & \Big(\lambda x.\nu(h(x),f,f_p), \mathsf{ap}_{\nu(-,f_p)}(h_p)\cdot\tau(\mathsf{glue}_{\Sigma X}(x_0),f_p)\Big) \\
=\ & (\lambda x.\nu(x,f\circ\Sigma h,f_p),\tau(\mathsf{glue}_{\Sigma Z}(z_0),f_p)) && (\Lambda(\rho_{\Sigma h},x,x_0)) \\
\equiv\ & \Phi(f\circ\Sigma h,f_p) \\
\equiv\ & \Phi((f,f_p)\circ\Sigma(h,h_p))
\end{aligned}
$$

Here, $\Lambda(\rho_{\Sigma h}, x, x_0)$ denotes the A-homotopy whose first component is the chain of paths

$$\begin{gathered} f_p^{-1} \cdot \mathsf{ap}_f(\mathsf{glue}_{\Sigma X}(h(x)) \cdot \mathsf{glue}_{\Sigma X}(h(x_0))^{-1}) \cdot f_p \\ \Big\|\, \textit{via } \rho_{\Sigma h}(x) \\ f_p^{-1} \cdot \mathsf{ap}_f(\mathsf{ap}_{\Sigma h}(\mathsf{glue}_Z(x)) \cdot \mathsf{ap}_{\Sigma h}(\mathsf{glue}_Z(x_0))^{-1}) \cdot f_p \\ \Big\|\, \mathsf{PI}(\mathsf{glue}_Z(x), \mathsf{glue}_Z(x_0)) \\ f_p^{-1} \cdot \mathsf{ap}_{f \circ \Sigma h}(\mathsf{glue}_Z(x) \cdot \mathsf{glue}_Z(x_0)^{-1}) \cdot f_p \end{gathered}$$

and whose second component is easily defined by path induction. With this definition of V_2, the 2-coherence diagram of Definition B.0.1 commutes. (This fact is formalized in [7, `HoTT-Agda/theorems/homotopy/SuspAdjointLoop.agda`].) We conclude that Σ preserves pointed colimits.

References

[1] J. F. Adams. *Stable homotopy and generalised homology.* University of Chicago Press, Chicago, Ill., 1974. Chicago Lectures in Mathematics.

[2] Jeremy Avigad, Krzysztof Kapulkin, and Peter LeFanu Lumsdaine. Homotopy limits in type theory. *Mathematical Structures in Computer Science*, 25(5):1040–1070, 2015. doi:10.1017/S0960129514000498.

[3] Ulrik Buchholtz, Tom de Jong, and Egbert Rijke. Epimorphisms and Acyclic Types in Univalent Foundations, 2024. arXiv:2401.14106.

[4] Ulrik Buchholtz and Kuen-Bang Hou (Favonia). Cellular Cohomology in Homotopy Type Theory. *Logical Methods in Computer Science*, Volume 16, Issue 2, 2020. doi:10.23638/LMCS-16(2:7)2020.

[5] Ulrik Buchholtz, Floris van Doorn, and Egbert Rijke. Higher Groups in Homotopy Type Theory. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '18, page 205–214, New York, NY, USA, 2018. Association for Computing Machinery. doi:10.1145/3209108.3209150.

[6] Evan Cavallo. Synthetic Cohomology in Homotopy Type Theory. Master's thesis, Carnegie Mellon University, 2015. URL: https://staff.math.su.se/evan.cavallo/works/thesis15.pdf.

[7] Perry Hart and Kuen-Bang Hou (Favonia). A formalized construction of coslice colimits. https://github.com/PHart3/colimits-agda/tree/v0.2.0, 2024.

[8] Daniel R. Licata and Guillaume Brunerie. A Cubical Approach to Synthetic Homotopy Theory. In *2015 30th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 92–103, 2015. doi:10.1109/LICS.2015.19.

[9] nLab authors. pointed abelian group. https://ncatlab.org/nlab/show/pointed+abelian+group, November 2024. Revision 3.

[10] Egbert Rijke. Introduction to Homotopy Type Theory, 2022. arXiv:2212.11082.

[11] Egbert Rijke, Michael Shulman, and Bas Spitters. Modalities in homotopy type theory. *Logical Methods in Computer Science*, Volume 16, Issue 1, January 2020. doi:10.23638/LMCS-16(1:2)2020.

[12] Egbert Rijke, Elisabeth Stenholm, Jonathan Prieto-Cubides, Fredrik Bakke, and others. The agda-unimath library. URL: https://github.com/UniMath/agda-unimath/.

[13] Kristina Sojakova, Floris van Doorn, and Egbert Rijke. Sequential Colimits in Homotopy Type Theory. In *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '20, page 845–858, 2020. doi:10.1145/3373718.3394801.

[14] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. https://homotopytypetheory.org/book, Institute for Advanced Study, 2013.