Cohomology

Combinatorial
Cellular & Abstract
Eilenberg-Steenrod

Ulrik Buchholtz and Favonia

Cohomology Groups

{ mappings from holes in a space }

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Cellular cohomology for CW complexes

Axiomatic Eilenberg-Steenrod cohomology

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Axiomatic Eilenberg-Steenrod cohomology

Dream: prove they are the same!

CW complexes

inductively-defined spaces

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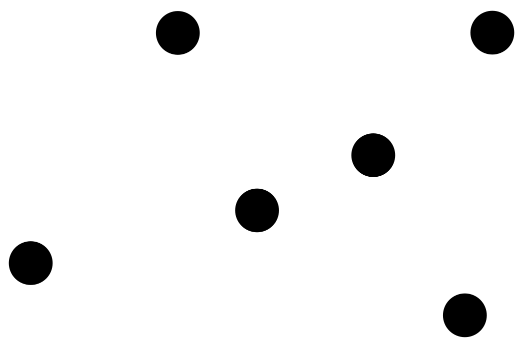
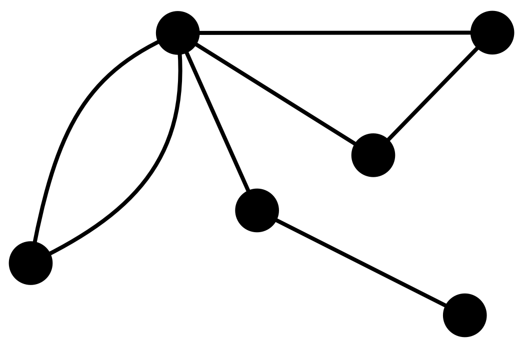


points

CW complexes

inductively-defined spaces



points
lines

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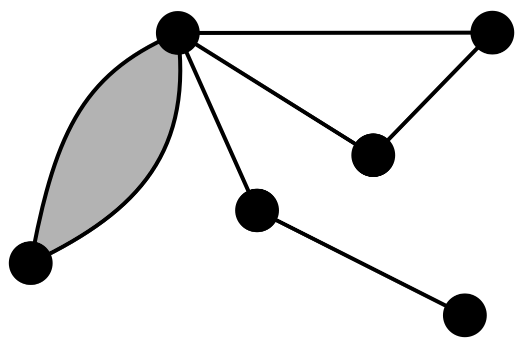

points

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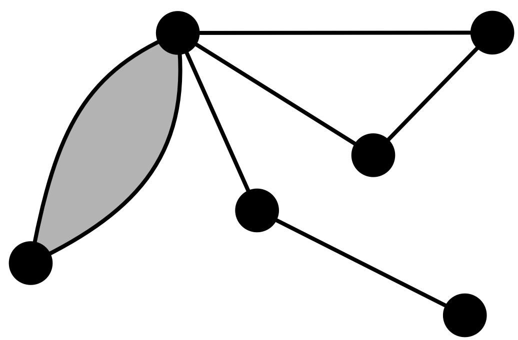

points
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(and more...)

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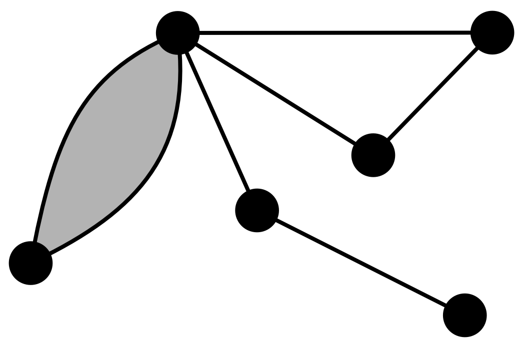

points
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faces
(and more...)

Specification: cells and how they attach

CW complexes

Sets of cells: A_n

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Attaching: $\alpha_{n+1} : A_{n+1} \times S^n \longrightarrow X_n$

X_n is the construction up to dim. n

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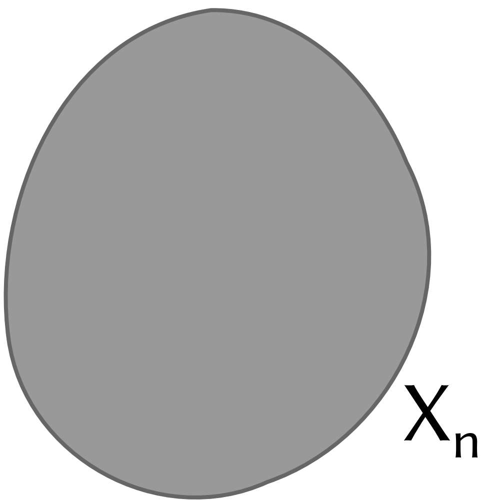

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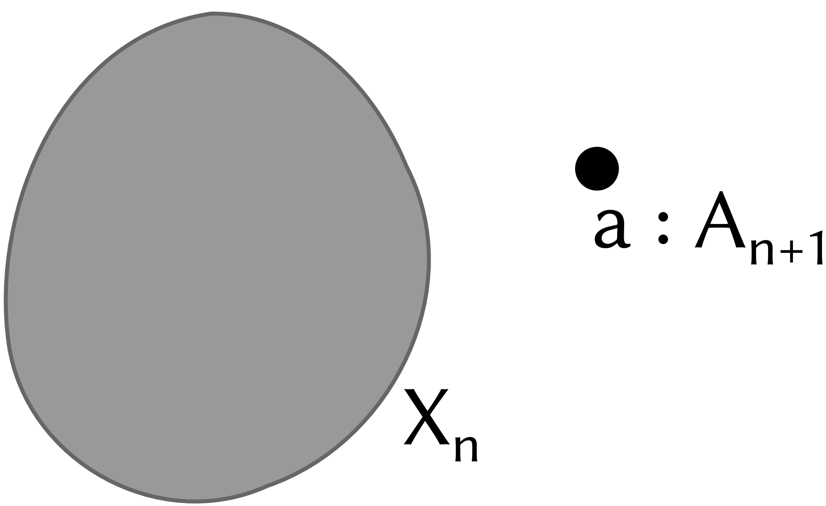

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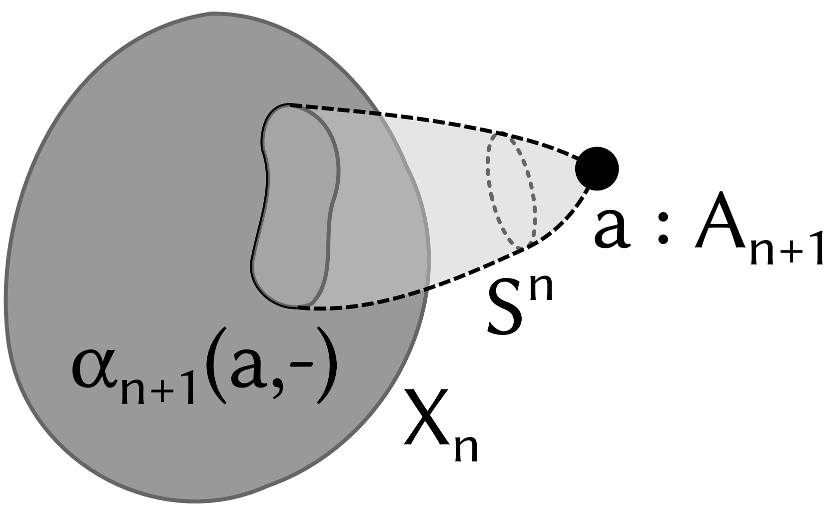

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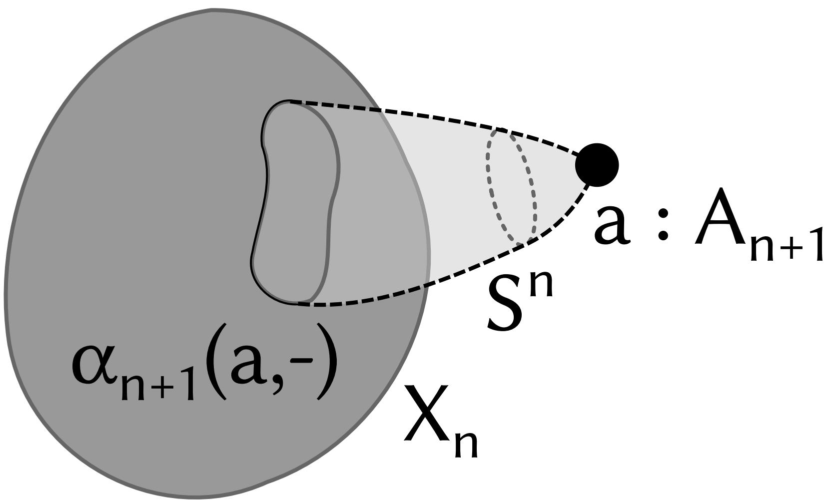

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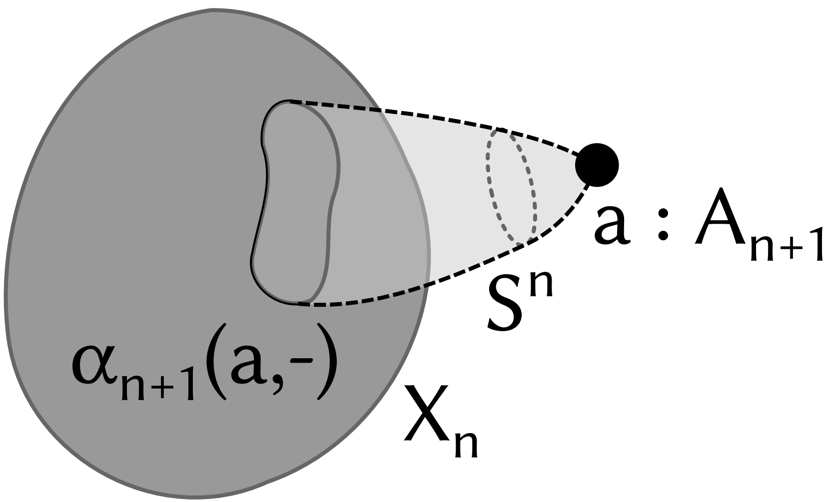

$X_0 := A_0$

$X_{n+1} :=$

$$\begin{tikzcd} A_{n+1} \times S^n \arrow[r] \arrow[d, "\alpha_{n+1}"'] & A_{n+1} \arrow[d] \\ X_n \arrow[r] & X_{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Cellular Cohomology

{ mappings from holes in a space }

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Cellular Homology

{ holes in a space }

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{ mappings from holes in a space }

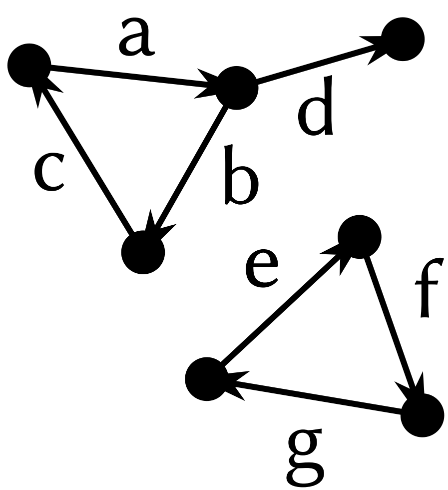
$$\uparrow \text{dualize}$$

Cellular Homology

{ holes in a space }

One-Dimensional Holes*

{ elements of $Z[A_1]$ forming cycles }



holes

*Holes are cycles in the classical homology theory

One-Dimensional Holes*

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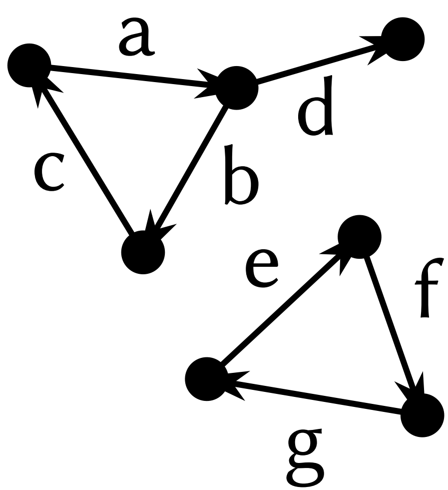

holes

a + b + c

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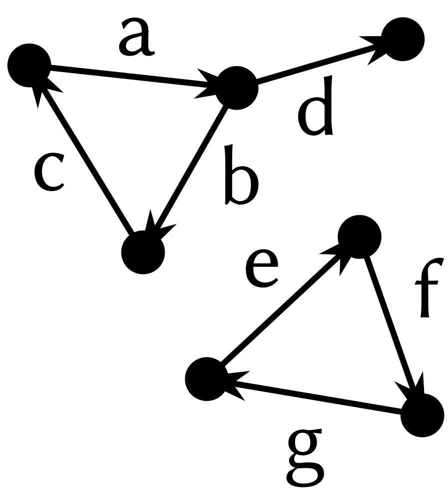

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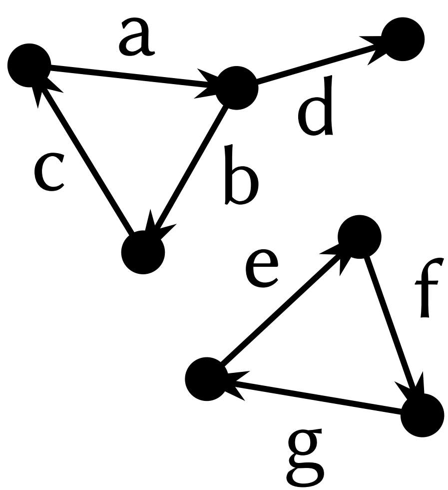

holes

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$a + b + c + e + g + f$

...

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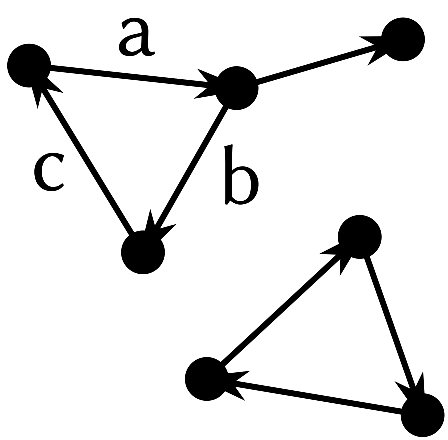

boundary function ∂

$\partial(x \xrightarrow{a} y) = y - x$

One-Dimensional Holes

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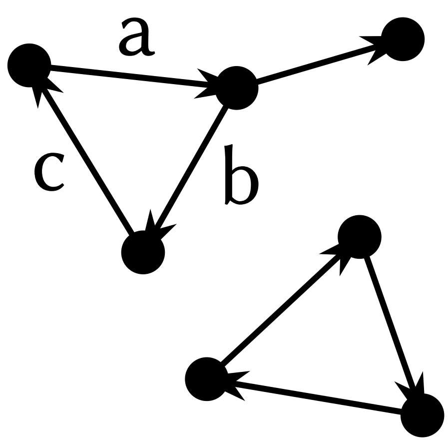

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$\boxed{\text{set of holes} = \text{kernel of } \partial}$

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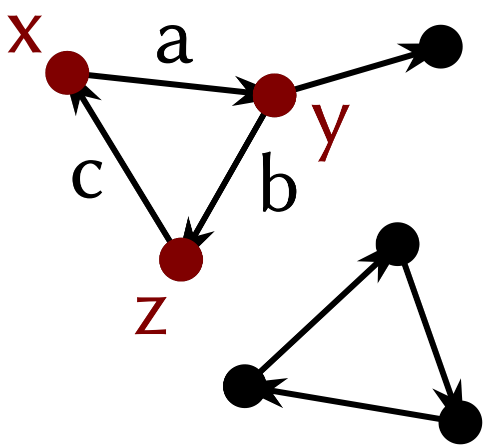

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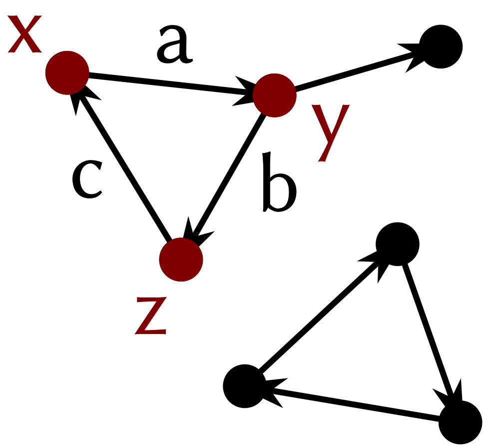


boundary function ∂

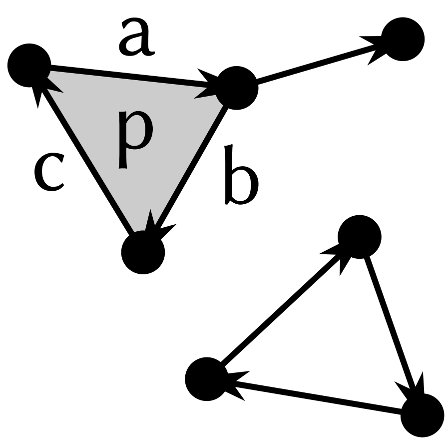
$$\partial(x \xrightarrow{a} y) = y - x$$

$$\partial(a+b+c) = (y - x) + (z - y) + (x - z) = 0$$

set of holes = kernel of ∂

First Homology Groups

{ unfilled one-dimensional holes }



First Homology Groups

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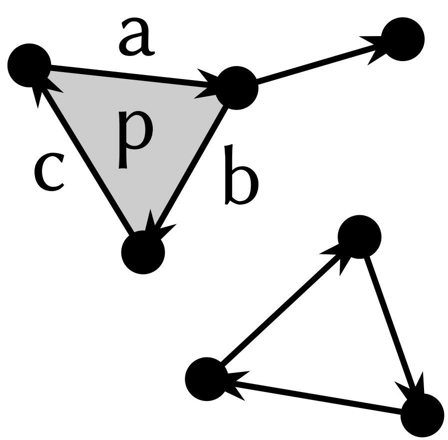
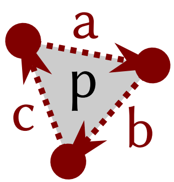


2-dim. boundary function ∂_2

$\partial_2($ 

$) = a + b + c$

First Homology Groups

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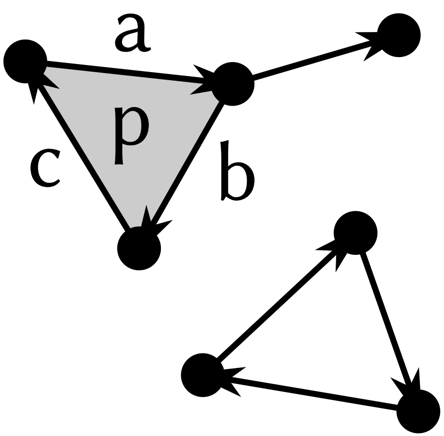

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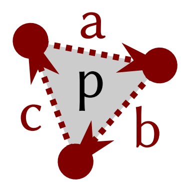
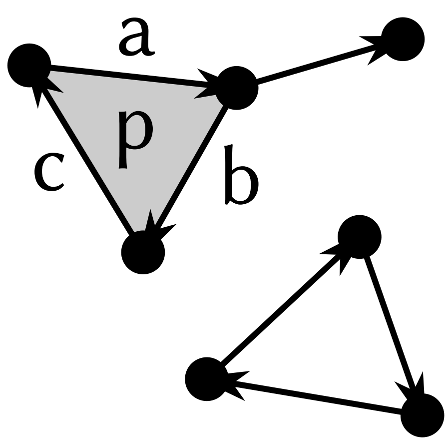
$\partial_2($ 

$) = a + b + c$

filled holes = image of ∂_2

First Homology Groups

{ unfilled one-dimensional holes }



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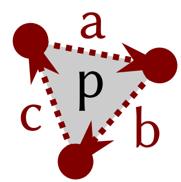
$\partial_2($ 

$) = a + b + c$

filled holes = image of ∂_2

$H_1(X)$:= kernel of ∂_1 / image of ∂_2

(unfilled holes) (all holes) (filled holes)

Homology Groups

{ unfilled holes }

$C_n := Z[A_n]$ formal sums of cells (chains)

Homology Groups

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$C_n := Z[A_n]$ formal sums of cells (chains)

$$\cdots \longrightarrow C_{n+2} \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \longrightarrow \cdots$$

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Cohomology Groups

$$\cdots \longrightarrow C_{n+2} \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \longrightarrow \cdots$$

Dualize by Hom(—, G). Let $C^n = \mathrm{Hom}(C_n, G)$.

$$\cdots \longleftarrow C^{n+2} \xleftarrow{\delta_{n+2}} C^{n+1} \xleftarrow{\delta_{n+1}} C^n \xleftarrow{\delta_n} C^{n-1} \xleftarrow{\delta_{n-1}} C^{n-2} \longleftarrow \cdots$$

Cohomology Groups

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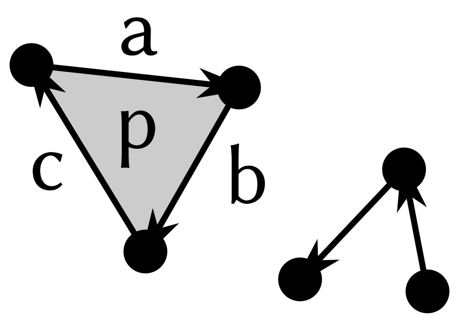
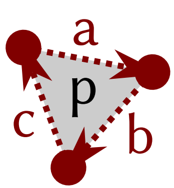
$H^n(X; G) :=$ kernel of δ_{n+1} / image of δ_n

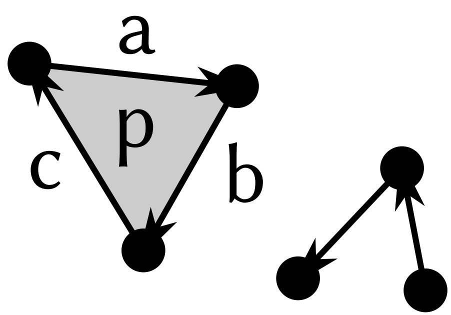
Higher-Dim. Boundary

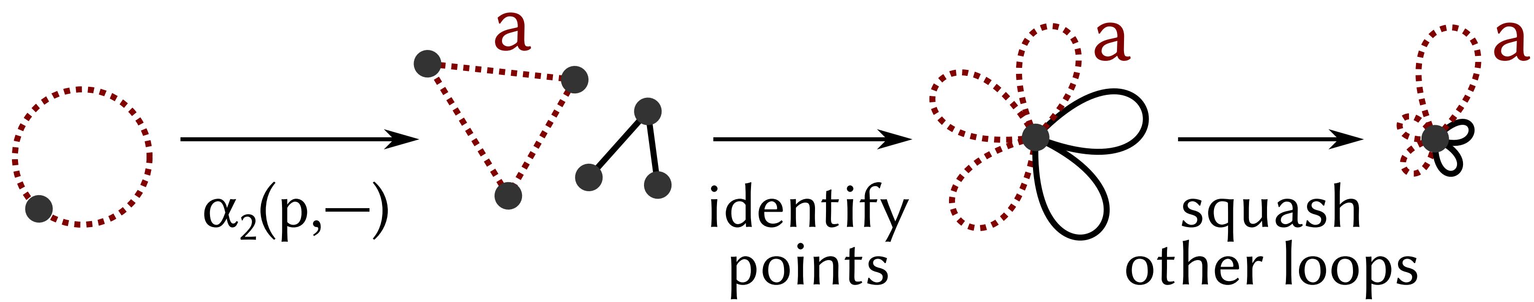


$\partial_2($  $) = a + b + c$

How to compute the coefficients from α_2?

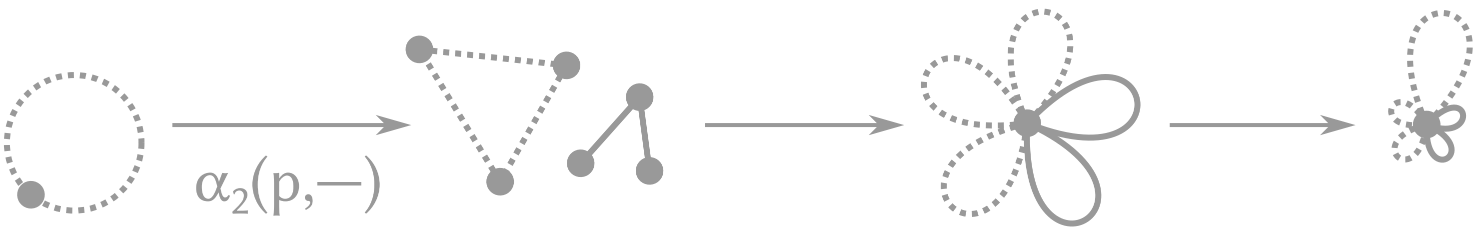
Higher-Dim. Boundary





coefficient = winding number of this map

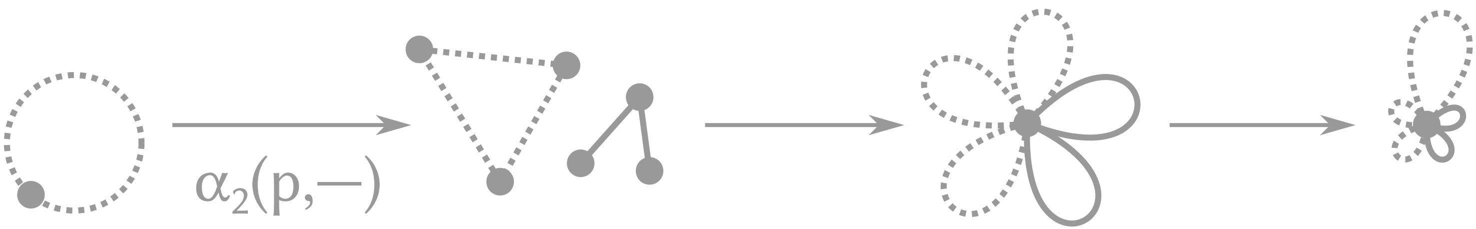
Higher-Dim. Boundary



$$S^n \xrightarrow[\alpha_{n+1}(p,—)]{} X_n \xrightarrow[\substack{\text{identify}\\ \text{lower structs.}}]{} X_n/X_{n-1} \simeq \bigvee S^n \xrightarrow[\text{squash}]{} S^n$$

coefficient = *degree* of this map

Higher-Dim. Boundary



$$S^n \xrightarrow[\alpha_{n+1}(p,-)]{} X_n \xrightarrow[\text{identify lower structs.}]{} X_n/X_{n-1} \simeq \bigvee S^n \xrightarrow[\text{squash}]{} S^n$$

coefficient = *degree* of this map

- squashing needs decidable equality
- linear sum needs closure-finiteness

Higher-Dim. Boundary

$$\begin{tikzcd}
 & & S^n \arrow[d, dashed] & \\
A_n \times S^{n-1} \arrow[r] \arrow[d] & A_n \arrow[d] & A_{n+1} \times S^n \arrow[r] \arrow[dl, dashed] & A_{n+1} \arrow[d] \\
X_{n-1} \arrow[r] \arrow[d] & X_n \arrow[rr] \arrow[d, dashed] \arrow[ul, phantom, "\ulcorner", very near start] & & X_{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \\
1 \arrow[r] & X_n/X_{n-1} \simeq \bigvee S^n \arrow[d, dashed] \arrow[ul, phantom, "\ulcorner", very near start] & & \\
 & S^n & &
\end{tikzcd}$$

Cohomology Groups

{ mappings from holes in a space }

Cellular cohomology for CW-complexes

$\mathbf{H^n(X; G)}$

Axiomatic Eilenberg-Steenrod cohomology

Dream: prove they are the same!

Eilenberg-Steenrod* cohomology

A family of functors $h^n(-)$:
pointed spaces $\longrightarrow$ abelian groups

* *Ordinary, reduced* cohomology theory

Eilenberg-Steenrod* cohomology

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1. $h^{n+1}(\mathrm{susp}(X)) \simeq h^n(X)$, natural in X

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$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ 1 \arrow[r] & \mathrm{Cof}_f \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

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2. $$\begin{tikzcd}
h^n(A) & & h^n(B) \arrow[ll, dotted] \\
A \arrow[r, "f"] \arrow[d] & B \arrow[d] & \\
1 \arrow[r] & \mathrm{Cof}_f \arrow[ul, phantom, "\ulcorner", very near start] & h^n(\mathrm{Cof}_f) \arrow[uu, dotted, "\text{exact!}"']
\end{tikzcd}$$

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3. $h^n(\bigvee_i X_i) \simeq \prod_i h^n(X_i)$
if the index type satisfies set-level AC

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Eilenberg-Steenrod* cohomology

A family of functors $h^n(-)$:
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$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ 1 \arrow[r] & \mathrm{Cof}_f \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}
\qquad
\begin{tikzcd} h^n(A) & h^n(B) \arrow[l, dotted] \\ & h^n(\mathrm{Cof}_f) \arrow[u, dotted] \end{tikzcd}$$
exact!

3. $h^n(\bigvee_i X_i) \simeq \prod_i h^n(X_i)$
if the index type satisfies set-level AC

4. $h^n(2)$ trivial for $n \neq 0$

* *Ordinary, reduced* cohomology theory

Cohomology Groups

{ mappings from holes in a space }

Cellular cohomology for CW-complexes	Axiomatic Eilenberg-Steenrod cohomology
$\mathbf{H^n(X;G)}$	$\mathbf{h^n(X)}$

Dream: prove they are the same!

Our Dream

$$h^n(X) \overset{?}{\simeq} H^n(X; h^0(2))$$

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$$\ker(\delta_{n+1})/\mathrm{im}(\delta_n)$$

Our Dream

$$\begin{array}{ccc}
h^n(X) & \overset{?}{\simeq} & H^n(X; h^0(2)) \\
\wr\,? & & \shortparallel \\
\ker(\delta'_{n+1})/\mathrm{im}(\delta'_n) & & \ker(\delta_{n+1})/\mathrm{im}(\delta_n)
\end{array}$$

Our Dream

$$\begin{array}{ccc} h^n(X) & \overset{?}{\simeq} & H^n(X; h^0(2)) \\ \wr\!\wr\ ? & & \ddot{\shortparallel} \\ \ker(\delta'_{n+1})/\mathrm{im}(\delta'_n) & & \ker(\delta_{n+1})/\mathrm{im}(\delta_n) \end{array}$$

1. Find δ' such that $h^n(X) \simeq \ker(\delta'_{n+1})/\mathrm{im}(\delta'_n)$
   done and fully mechanized in Agda

2. Show δ and δ' are equivalent
   domains and codomains are isomorphic
   commutativity in progress

Our Dream: Step 1 (done!)

For any pointed CW-complex X where

1. all cell sets A_n satisfy set-level AC and
2. the point of A_0 is *separable* (pt = x is decidable)

there exist homomorphisms δ'

$$\cdots \xleftarrow{} D^{n+2} \xleftarrow{\delta'_{n+2}} D^{n+1} \xleftarrow{\delta'_{n+1}} D^{n} \xleftarrow{\delta'_{n}} D^{n-1} \xleftarrow{\delta'_{n-1}} D^{n-2} \xleftarrow{} \cdots$$

such that

$$h^n(X) \simeq \text{kernel of } \delta'_{n+1} \,/\, \text{image of } \delta'_n$$

Important Lemmas for Step 1

Long exact sequenses

$$
\begin{tikzcd}
h^n(A) \arrow[rrrdd, dotted, "n{+}{+}"'] & & & h^n(B) \arrow[lll, dotted] \\
& A \arrow[r, "f"] \arrow[d] & B \arrow[d] & \\
& 1 \arrow[r] & \mathrm{Cof}_f \arrow[ul, phantom, "\ulcorner", very near start] & h^n(\mathrm{Cof}_f) \arrow[uu, dotted]
\end{tikzcd}
$$

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Long exact sequenses

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h^n(A) \arrow[ddrr, dotted, "n{+}{+}"'] & & h^n(B) \arrow[ll, dotted] \\
A \arrow[r, "f"] \arrow[d] & B \arrow[d] & \\
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\end{tikzcd}$$

Wedges of cells

$h^m(X_n/X_{n-1}) \simeq \hom(Z[A_n], h^0(2))$
when m = n or trivial otherwise

$h^m(X_0) \simeq \hom(Z[A_0\backslash\{pt\}], h^0(2))$
when m = 0 or trivial otherwise

Important Lemmas for Step 1

Long exact sequenses

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ 1 \arrow[r] & \mathrm{Cof}_f \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

$$\begin{tikzcd} h^n(A) \arrow[dr, dotted, "n{+}{+}"'] & & h^n(B) \arrow[ll, dotted] \\ & & h^n(\mathrm{Cof}_f) \arrow[u, dotted] \end{tikzcd}$$

Wedges of cells

$$\left.\begin{aligned} & h^m(X_n/X_{n-1}) \simeq \hom(Z[A_n], h^0(2)) \\ & \text{when } m = n \text{ or trivial otherwise} \\ & h^m(X_0) \simeq \hom(Z[A_0\backslash\{pt\}], h^0(2)) \\ & \text{when } m = 0 \text{ or trivial otherwise} \end{aligned}\right\} \text{trivial if } m \neq n$$

Ultimate Cofiber Diagram

$$
\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_{n-1} \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & X_{n+1} \arrow[d] & \cdots \\
1 \arrow[r] & \cdots \arrow[r] & X_{n-1/0} \arrow[r] \arrow[d] & X_{n/0} \arrow[r] \arrow[d] & X_{n+1/0} \arrow[d] & \cdots \\
 & & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & & 1 \arrow[r] & X_{n/n-1} \arrow[r] \arrow[d] & X_{n+1/n-1} \arrow[d] & \cdots \\
 & & & 1 \arrow[r] & X_{n+1/n} & \cdots
\end{tikzcd}
$$

$\boxed{X_{n/m} := X_n/X_m}$

Ultimate Cofiber Diagram

$$
\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_{n-1} \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & X_{n+1} \arrow[d] & \cdots \\
1 \arrow[r] & \cdots \arrow[r] & X_{n-1/0} \arrow[r] \arrow[d] & X_{n/0} \arrow[r] \arrow[d] & X_{n+1/0} \arrow[d] & \cdots \\
 & & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & & 1 \arrow[r] & X_{n/n-1} \arrow[r] \arrow[d] & X_{n+1/n-1} \arrow[d] & \cdots \\
 & & & 1 \arrow[r] & X_{n+1/n} & \cdots
\end{tikzcd}
$$

$\boxed{X_{n/m} := X_n/X_m}$

Plan:

Obtain long exact sequences and use group-theoretic magic

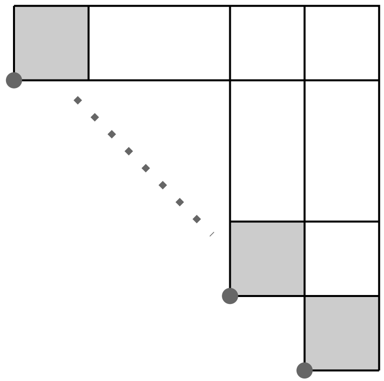
Ultimate Cofiber Diagram

$$
\begin{tikzcd}
X_0 \arrow[r] & \cdots \arrow[r] & X_{n-1} \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & X_{n+1} \arrow[d] & \cdots \\
1 \arrow[r] & \cdots \arrow[r] & X_{n-1/0} \arrow[r] \arrow[d] \arrow[ddrr, dotted] & X_{n/0} \arrow[r] \arrow[d] & X_{n+1/0} \arrow[d] \arrow[ll, dotted, bend right] & \cdots \\
 & & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
 & & 1 \arrow[r] & X_{n/n-1} \arrow[r] \arrow[d] & X_{n+1/n-1} \arrow[d] \arrow[uu, dotted, bend right] & \cdots \\
 & & & 1 \arrow[r] & X_{n+1/n} & \cdots
\end{tikzcd}
$$

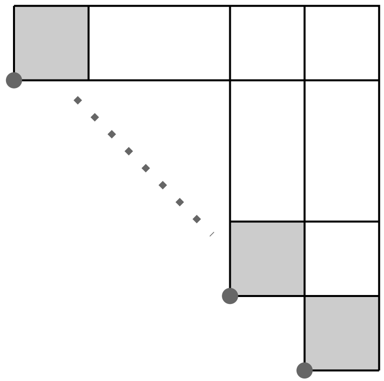
$\boxed{X_{n/m} := X_n/X_m}$

Plan:

Obtain long exact sequences and use group-theoretic magic

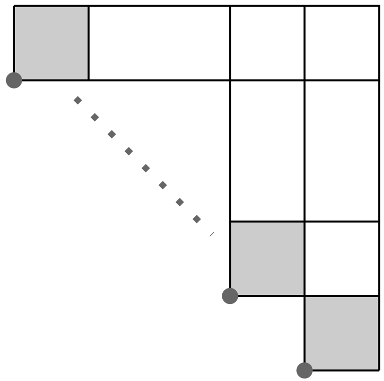


$$\begin{tikzcd} X_{n/n-1} \arrow[r] \arrow[d] & X_{n+1/n-1} \arrow[d] \\ 1 \arrow[r] & X_{n+1/n} \end{tikzcd}$$



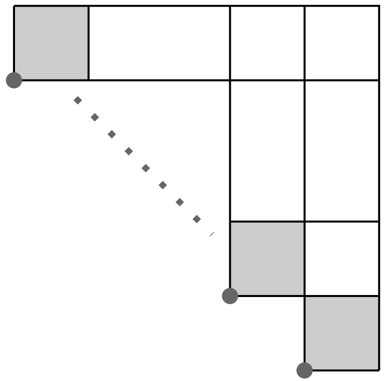
$$\begin{tikzcd}
X_{n/n-1} \arrow[r] \arrow[d] \arrow[dr, dashed] & X_{n+1/n-1} \arrow[d] \\
1 \arrow[r] & X_{n+1/n}
\end{tikzcd}$$

$$\begin{tikzcd}
h^n(X_{n+1/n}) \arrow[r] & h^n(X_{n+1/n-1}) \arrow[r] & h^n(X_{n/n-1}) \arrow[dll, dashed] \\
h^{n+1}(X_{n+1/n}) \arrow[r] & h^{n+1}(X_{n+1/n-1}) \arrow[r] & h^{n+1}(X_{n/n-1})
\end{tikzcd}$$



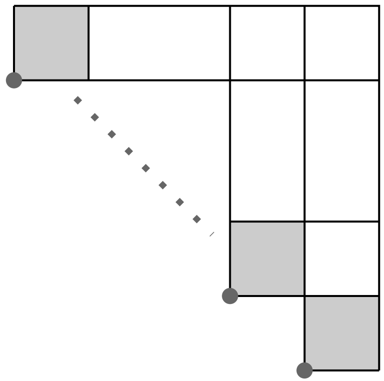
$$\begin{tikzcd}
X_{n/n-1} \arrow[r] \arrow[d] \arrow[dr, dashed] & X_{n+1/n-1} \arrow[d] \\
1 \arrow[r] & X_{n+1/n}
\end{tikzcd}$$

$$\begin{tikzcd}
h^n(X_{n+1/n}) \arrow[r] & h^n(X_{n+1/n-1}) \arrow[r] & h^n(X_{n/n-1}) \arrow[dll, dashed, "\text{our choice of } \delta'"] \\
h^{n+1}(X_{n+1/n}) \arrow[r] & h^{n+1}(X_{n+1/n-1}) \arrow[r] & h^{n+1}(X_{n/n-1})
\end{tikzcd}$$



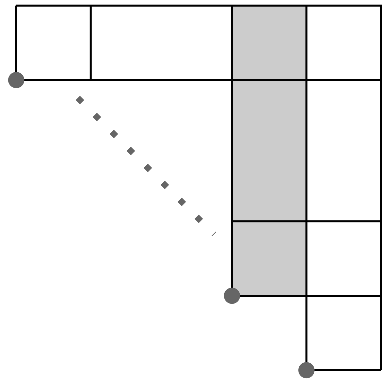
$$\begin{tikzcd}
X_{n/n-1} \arrow[r] \arrow[d] \arrow[dr, dashed] & X_{n+1/n-1} \arrow[d] \\
1 \arrow[r] & X_{n+1/n}
\end{tikzcd}$$

$$\begin{tikzcd}
\text{trivial} & & \\
h^n(X_{n+1/n}) \arrow[r] & h^n(X_{n+1/n-1}) \arrow[r] & h^n(X_{n/n-1}) \arrow[dll, dashed, bend right, "\text{our choice of } \delta'"'] \\
h^{n+1}(X_{n+1/n}) \arrow[r] & h^{n+1}(X_{n+1/n-1}) \arrow[r] & h^{n+1}(X_{n/n-1}) \\
& & \text{trivial}
\end{tikzcd}$$

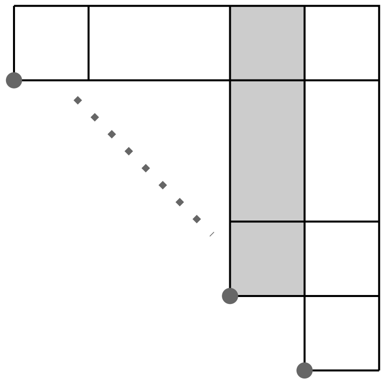


$$
\begin{tikzcd}
X_{n/n-1} \arrow[r] \arrow[d] \arrow[dr, dashed] & X_{n+1/n-1} \arrow[d] \\
1 \arrow[r] & X_{n+1/n}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\text{trivial} & \ker(\delta') \arrow[d, phantom, "\cong" rotate=90] & \\
h^n(X_{n+1/n}) \arrow[r] & h^n(X_{n+1/n-1}) \arrow[r, "\text{inj}"] & h^n(X_{n/n-1}) \arrow[dll, dashed, bend left, "\text{our choice of } \delta'"] \\
h^{n+1}(X_{n+1/n}) \arrow[r, "\text{surj}"'] & h^{n+1}(X_{n+1/n-1}) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & h^{n+1}(X_{n/n-1}) \\
 & \operatorname{coker}(\delta') & \text{trivial}
\end{tikzcd}
$$

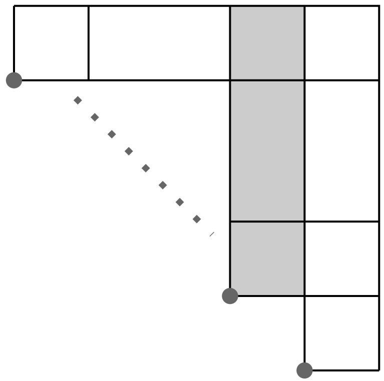


$$\begin{tikzcd} X_m \arrow[r] \arrow[d] & X_{m+1} \arrow[d] \\ 1 \arrow[r] & X_{m+1/m} \end{tikzcd}$$



$$\begin{tikzcd} X_m \arrow[r] \arrow[d] \arrow[dr, dashed] & X_{m+1} \arrow[d] \\ 1 \arrow[r] & X_{m+1/m} \end{tikzcd}$$

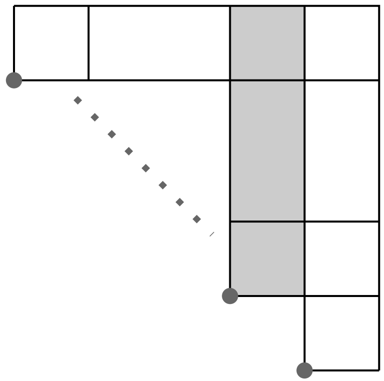
$$h^n(X_{m+1/m}) \longrightarrow h^n(X_{m+1}) \longrightarrow h^n(X_m) \longrightarrow h^{n+1}(X_{m+1/m})$$



$$\begin{tikzcd} X_m \arrow[r] \arrow[d] \arrow[dr, dashed] & X_{m+1} \arrow[d] \\ 1 \arrow[r] & X_{m+1/m} \end{tikzcd}$$

$$h^n(X_{m+1/m}) \longrightarrow h^n(X_{m+1}) \longrightarrow h^n(X_m) \longrightarrow h^{n+1}(X_{m+1/m})$$

If $n \neq m, m+1$, both ends trivial, $h^n(X_{m+1}) \simeq h^n(X_m)$

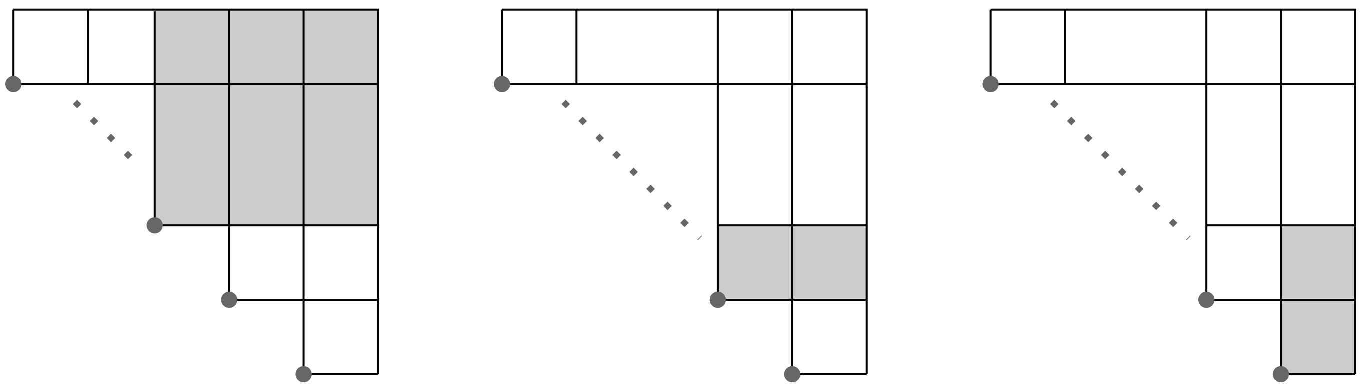


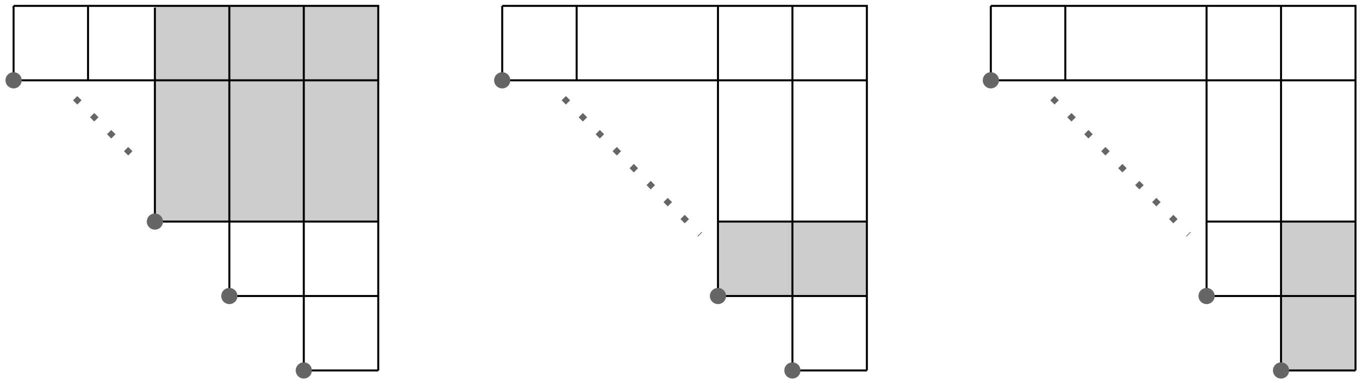
$$\begin{tikzcd} X_m \arrow[r] \arrow[d] & X_{m+1} \arrow[d] \\ 1 \arrow[r] & X_{m+1/m} \end{tikzcd}$$

$$h^n(X_{m+1/m}) \longrightarrow h^n(X_{m+1}) \longrightarrow h^n(X_m) \longrightarrow h^{n+1}(X_{m+1/m})$$

If $n \neq m, m+1$, both ends trivial, $h^n(X_{m+1}) \simeq h^n(X_m)$

three possible values
$$\left\{ \begin{array}{l} h^n(X_{n-1}) \simeq h^n(X_{n-2}) \simeq \cdots \simeq h^n(X_0), \text{ trivial} \\ h^n(X_n) \\ h^n(X_{n+1}) \simeq h^n(X_{n+2}) \simeq \cdots \simeq h^n(X) \end{array} \right.$$





$$\operatorname{coker}(\delta'_n) \simeq$$

$$\begin{tikzcd}
h^n(X_{n/n-2}) & h^n(X_{n+1/n-2}) \arrow[l] \\
h^n(X_{n/n-1}) \arrow[u, "\text{eq. class}"] & h^n(X_{n+1/n-1}) \arrow[l, "\text{inject}"'] \arrow[u]
\end{tikzcd}$$

$$h^n(X_{n+1/n-2}) \simeq h^n(X)$$

$$h^n(X_{n+1/n-1}) \simeq \ker(\delta'_{n+1})$$

$$
\begin{tikzcd}
\mathrm{coker}(\delta'_n) & h^n(X) \arrow[l, tail] \\
h^n(X_{n/n-1}) \arrow[u, two heads, "\text{eq. class}"] & \mathrm{ker}(\delta'_{n+1}) \arrow[l, tail, "\text{inject}"'] \arrow[u, two heads]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{coker}(\delta'_n) & \color{red}{h^n(X)} \arrow[l, tail] \\
h^n(X_{n/n-1}) \arrow[u, two heads, "\text{eq. class}"] & \ker(\delta'_{n+1}) \arrow[l, tail, "\text{inject}"'] \arrow[u, two heads]
\end{tikzcd}
$$

Using group-theoretic magic...

$$h^n(X) \simeq \ker(\delta'_{n+1})/\mathrm{im}(\delta'_n)$$

Our Dream (updated)

$$h^n(X) \overset{?}{\simeq} H^n(X; h^0(2))$$

$$\begin{array}{ccc} h^n(X) & \overset{?}{\simeq} & H^n(X; h^0(2)) \\ \wr\| & & \| \\ \ker(\delta'_{n+1})/\mathrm{im}(\delta'_n) & & \ker(\delta_{n+1})/\mathrm{im}(\delta_n) \end{array}$$

1. Find δ' such that $h^n(X) \simeq \ker(\delta'_{n+1})/\mathrm{im}(\delta'_n)$
2. Show δ and δ' are equivalent
   domains and codomains are isomorphic
   commutativity in progress

Cohomology Groups

Cellular coh.
for pointed
CW complexes

Ordinary reduced
cohomology
theories

Dream: prove they give the same groups

We made an important step in proving it