Cohomology Groups

\{ mappings from holes in a space \}
Cohomology Groups

\{ mappings from holes in a space \}

Cellular cohomology for CW complexes

Axiomatic Eilenberg-Steenrod cohomology

Goal: prove they are the same!
CW complexes
inductively-defined spaces
CW complexes
inductively-defined spaces
points
CW complexes
inductively-defined spaces

points
lines
CW complexes
inductively-defined spaces

points
lines
faces
(and more...)

Data: cells and how they attach
CW complexes

Sets of cell indices: $A_n$

Attaching: $\alpha_{n+1}: A_{n+1} \times S^n \rightarrow X_n$

$X_n$ is the construction up to dim. $n$

$X_0 := A_0$

$X_{n+1} :=$

$A_{n+1} \times S^n \rightarrow A_{n+1}$

$\alpha_{n+1}$

$\Gamma$

$X_n \rightarrow X_{n+1}$
CW complexes

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$X_0 := A_0$

$X_{n+1} :=$

\[ A_{n+1} \times S^n \longrightarrow A_{n+1} \]

\[ \alpha_{n+1} \downarrow \quad \Gamma \downarrow \]

\[ X_n \quad \longrightarrow \quad X_{n+1} \]

$X_n$
CW complexes

Sets of cell indices: $A_n$

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$A_{n+1} \times S^n \to A_{n+1}$

$\alpha_{n+1} \downarrow$

$\Gamma$

$X_n \to X_{n+1}$
**CW complexes**

Sets of cell indices: $A_n$

Attaching: $\alpha_{n+1} : A_{n+1} \times S^n \rightarrow X_n$

$X_n$ is the construction up to dim. $n$

\[ X_0 := A_0 \]

\[ X_{n+1} := \]

\[ A_{n+1} \times S^n \rightarrow A_{n+1} \]

\[ \alpha_{n+1} \]

\[ X_n \]

\[ \Gamma \]

\[ X_{n+1} \]
Cellular Cohomology
\{ mappings from holes in a space \}

Cellular Homology
\{ holes in a space \}

dualize
One-Dimensional Holes*  
{ elements of $\mathbb{Z}[A_1]$ forming cycles }  

\[ a + b + c \]  
\[ -a - b - c \]  
\[ a + b + c + e + g + f \]  
...  

*Holes are cycles in the classical homology theory
One-Dimensional Holes

\{ \text{elements of } \mathbb{Z}\left[ A_1 \right] \text{ forming cycles} \}

boundary function $\partial$

$\partial(\begin{array}{c} a \\ x \end{array}) = y - x$

$\partial(a+b+c) = (y - x) + (z - y) + (x - z) = 0$

set of holes = kernel of $\partial$
First Homology Groups
{ unfilled one-dimensional holes }

\[ \partial_2(a + b + c) = \text{filled holes} = \text{image of } \partial_2 \]
First Homology Groups

\{ \text{unfilled one-dimensional holes} \}

\[ \partial_2( ) = a + b + c \]

2-dim. boundary function \( \partial_2 \)

\( \partial_2( ) = a + b + c \)

filled holes = image of \( \partial_2 \)

\( H_1(X) := \text{kernel of } \partial_1 / \text{image of } \partial_2 \)

(unfilled holes) \hspace{1cm} (all holes) \hspace{1cm} (filled holes)
Homology Groups

\{ \text{unfilled holes} \}

\[ C_n := \mathbb{Z}[A_n] \text{ formal sums of cells (chains)} \]

\[ \cdots \rightarrow C_{n+2} \rightarrow C_{n+1} \rightarrow C_n \rightarrow C_{n-1} \rightarrow C_{n-2} \rightarrow \cdots \]

\[ H_n(X) := \text{kernel of } \partial_n / \text{image of } \partial_{n+1} \]
Cohomology Groups

\[ \ldots \rightarrow C_{n+2} \rightarrow C_{n+1} \rightarrow C_n \rightarrow C_{n-1} \rightarrow C_{n-2} \rightarrow \ldots \]

Dualize by \( \text{Hom}(-, G) \). Let \( C^n = \text{Hom}(C_n, G) \).

\[ \ldots \leftarrow C^{n+2} \leftarrow C^{n+1} \leftarrow C^n \leftarrow C^{n-1} \leftarrow C^{n-2} \leftarrow \ldots \]

\( H^n(X; G) := \text{kernel of } \delta_{n+1} / \text{image of } \delta_n \)
2-Dimensional Boundary

\[ \partial_2( p ) = a + b + c \]

How to compute the coefficients from \( \alpha_2 \)?
2-Dimensional Boundary

\[ \alpha_2(p, -) \]

identify points

squash other loops

\text{coefficient} = \text{winding number of this map}

(can be generalized to higher dimensions)
Cohomology Groups
{ mappings from holes in a space }

Cellular cohomology for CW-complexes

$H^n(X; G)$

Axiomatic Eilenberg-Steenrod cohomology

Prove they are the same!
Eilenberg-Steenrod* cohomology

A family of functors $h^n(\_\_)$: pointed spaces $\rightarrow$ abelian groups

*ordinary, reduced
Eilenberg-Steenrod* cohomology

A family of functors $h^n(\_\_)$:
pointed spaces $\rightarrow$ abelian groups

1. $h^{n+1}(\text{susp}(X)) \cong h^n(X)$, natural in $X$

*ordinary, reduced
Eilenberg-Steenrod* cohomology

A family of functors $h^n(-)$:
pointed spaces $\rightarrow$ abelian groups

1. $h^{n+1}(\text{susp}(X)) \cong h^n(X)$, natural in $X$

2. $A \xrightarrow{f} B$

$\downarrow \quad \downarrow$

$1 \xrightarrow{} \text{Cof}_f$

*ordinary, reduced
Eilenberg-Steenrod* cohomology

A family of functors $h^n(-)$:
pointed spaces $\rightarrow$ abelian groups

1. $h^{n+1}(\text{susp}(X)) \cong h^n(X)$, natural in $X$

2. $h^n(A) \xleftarrow{f} h^n(B)$

$A \xrightarrow{f} B \xrightarrow{\Gamma} \text{Cof}_f \xrightarrow{h^n} h^n(\text{Cof}_f)$

*ordinary, reduced
Eilenberg-Steenrod* cohomology

A family of functors $h^n(\_\_\):$
pointed spaces $\rightarrow$ abelian groups

1. $h^{n+1}(\text{susp}(X)) \cong h^n(X)$, natural in $X$

2. $h^n(A) \leftarrow h^n(B)$

3. $h^n(\bigvee_i X_i) \cong \prod_i h^n(X_i)$
if the index type is nice enough**

*ordinary, reduced

**see our paper
Eilenberg-Steenrod\(^*\) cohomology

A family of functors \(h^n(\_):\) pointed spaces \(\rightarrow\) abelian groups

1. \(h^{n+1}(\text{susp}(X)) \cong h^n(X),\) natural in \(X\)

2. \(h^n(A) \xrightarrow{f} h^n(B)\) if \(f\) is exact!

3. \(h^n(\bigvee_i X_i) \cong \prod_i h^n(X_i)\) if the index type is nice enough\(^**\)

4. \(h^n(2)\) trivial for \(n \neq 0\)

\(^*\) ordinary, reduced \(^**\) see our paper
Cohomology Groups
\{ mappings from holes in a space \}

\[ H^n(X; G) \quad h^n(X) \]

Cellular cohomology for CW-complexes
Axiomatic Eilenberg-Steenrod cohomology

Prove they are the same!
Proof Plan

$H^n(X; h^0(2)) \cong? h^n(X)$
Proof Plan

\[ H^n(X; h^0(2)) \cong \ker(\delta_{n+1})/\text{im}(\delta_n) \]\n
1. Find \( \delta' \) such that \( h^n(X) \cong \ker(\delta'_{n+1})/\text{im}(\delta'_n) \)
Proof Plan

\[ H^n(X; h^0(2)) \cong h^n(X) \]

\[ \cong \frac{\ker(\delta_{n+1})}{\text{im}(\delta_n)} \cong \frac{\ker(\delta'_{n+1})}{\text{im}(\delta'_n)} \]

1. Find \( \delta' \) such that \( h^n(X) \cong \frac{\ker(\delta'_{n+1})}{\text{im}(\delta'_n)} \)

2. Show \( \delta \) and \( \delta' \) are equivalent
Proof Plan

\[ H^n(X; h^0(2)) \cong h^n(X) \]

\[ \ker(\delta_{n+1})/\text{im}(\delta_n) \cong \ker(\delta'_{n+1})/\text{im}(\delta'_n) \]

\[ \delta \cong \delta' \]

1. Find \( \delta' \) such that \( h^n(X) \cong \ker(\delta'_{n+1})/\text{im}(\delta'_n) \)

2. Show \( \delta \) and \( \delta' \) are equivalent

As usual, fully mechanized in Agda!
Step 1: Reverse Engineering

For any theory \( h \), finite pointed CW-complex \( X \), there exist homomorphisms \( \delta' \)

\[
\delta'_{n+2} \leftarrow \delta'_{n+1} \leftarrow \delta'_n \leftarrow \delta'_{n-1} \leftarrow \ldots
\]

\[
\underset{\text{such that}}{\text{such that}}
\]

\[ h^n(X) \cong \text{kernel of } \delta'_{n+1} / \text{image of } \delta'_n \]
Important Lemmas for Step 1

Long exact sequences

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
1 & \xrightarrow{\text{Cof}_f} & \text{Cof}_f
\end{array}
\]
Important Lemmas for Step 1

Long exact sequences

\[ h^n(A) \xleftarrow{f} h^n(B) \xrightarrow{n++} h^n(C_{of}) \]

\[ h^{n+1}(A) \xrightarrow{n} h^{n+1}(B) \xrightarrow{n+1}(C_{of}) \]
Important Lemmas for Step 1

Long exact sequences

\[ h^n(A) \xrightarrow{f} h^n(B) \]

\[ A \xrightarrow{n++} B \]

\[ 1 \xrightarrow{} \text{Cof}_f h^n(\text{Cof}_f) \]

Wedges of cells

\[ h^m(X_n/X_{n-1}) \cong \text{hom}(\mathbb{Z}[A_n], h^0(2)) \]

when \( m = n \) or trivial otherwise

\[ h^m(X_0) \cong \text{hom}(\mathbb{Z}[A_0\setminus\{\text{pt}\}], h^0(2)) \]

when \( m = 0 \) or trivial otherwise

\[ \text{trivial if } m \neq n \]
Ultimate Cofiber Diagram

\[ X_0 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n \longrightarrow X_{n+1} \longrightarrow \cdots \]

\[ 1 \longrightarrow \cdots \longrightarrow X_{n-1/0} \longrightarrow X_{n/0} \longrightarrow X_{n+1/0} \longrightarrow \cdots \]

\[ X_{n/m} := X_n/X_m \]

\[ 1 \longrightarrow X_{n/n-1} \longrightarrow X_{n+1/n-1} \longrightarrow \cdots \]

\[ 1 \longrightarrow X_{n+1/n} \longrightarrow \cdots \]
Ultimate Cofiber Diagram

\[ \begin{array}{cccccccc}
X_0 & \rightarrow & \cdots & \rightarrow & X_{n-1} & \rightarrow & X_n & \rightarrow & X_{n+1} \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \cdots & \rightarrow & X_{n-1/0} & \rightarrow & X_{n/0} & \rightarrow & X_{n+1/0} \\
\downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
& & & & 1 & \rightarrow & \cdots & \rightarrow & X_{n/n-1} \\
& & & & \downarrow & & \downarrow & & \downarrow \\
& & & & 1 & \rightarrow & \cdots & \rightarrow & X_{n+1/n} \\
\end{array} \]

\( X_{n/m} := \frac{X_n}{X_m} \)

Plan:
Obtain long exact sequences and use group-theoretic magic
Ultimate Cofiber Diagram

\[ X_0 \rightarrow \cdots \rightarrow X_{n-1} \rightarrow X_n \rightarrow X_{n+1} \rightarrow \cdots \]

\[ 1 \rightarrow \cdots \rightarrow X_{n-1/0} \rightarrow X_{n/0} \rightarrow X_{n+1/0} \rightarrow \cdots \]

\[ X_{n/m} := X_n/X_m \]

**Plan:**

Obtain long exact sequences and use group-theoretic magic
\[ h^n(X) \approx \ker(\delta'_{n+1})/\text{im}(\delta'_n) \]
Proof Plan (updated)

\[ H^n(X; h^0(2)) \cong h^n(X) \]

\[
\begin{array}{ccc}
\text{ker}(\delta_{n+1})/\text{im}(\delta_n) & \cong & \text{ker}(\delta_{n+1}')/\text{im}(\delta_n') \\
\end{array}
\]

1. Find \( \delta' \) such that \( h^n(X) \cong \ker(\delta_{n+1}')/\text{im}(\delta_n') \)

2. Show \( \delta \) and \( \delta' \) are equivalent
Step 2: Calculation

\[ \delta_{n+2} \leftarrow C_{n+2} \leftarrow C_{n+1} \leftarrow C_n \leftarrow C_{n-1} \leftarrow C_{n-2} \leftarrow \ldots \]

\[ \delta'_{n+2} \leftarrow D_{n+2} \leftarrow D_{n+1} \leftarrow D_n \leftarrow D_{n-1} \leftarrow D_{n-2} \leftarrow \ldots \]

The n=0 case \((C^0 \simeq D^0)\) is interesting
Summary

Cellular cohomology groups \cong \text{Ordinary reduced cohomology groups (finite)}
Summary

Cellular cohomology groups \( \cong \) Ordinary reduced cohomology groups

(finite)

To-Do

Infinity: colimits
Homology \( \Rightarrow \) Poincaré duality, ...
Parametrization \( \Rightarrow \) non-orientability, ...
Higher-Dim. Boundary

\[ S^n \xrightarrow{\alpha_{n+1}(p,-)} X_n \xrightarrow{\alpha_2(p,-)} X_n/X_{n-1} \cong \bigvee S^n \xrightarrow{\text{squash}} S^n \]

- \(\alpha_{n+1}(p,-)\) identify lower structs.
- \(\alpha_2(p,-)\)

Coefficient = degree of this map
Higher-Dim. Boundary

$S^n \xrightarrow{\alpha_2(p,-)} X_n \xrightarrow{\alpha_{n+1}(p,-)} X_n/X_{n-1} \cong \bigvee S^n \xrightarrow{\text{squash}} S^n$

identify lower structs.

coefficient = degree of this map

- squashing needs decidable equality
- linear sum needs closure-finiteness (free for finite cases)
Higher-Dim. Boundary

\[
\begin{align*}
A_n \times S^{n-1} &\longrightarrow A_n \\
S^n &\downarrow \\
X_{n-1} &\longrightarrow X_n \\
&\quad \searrow S^n
\end{align*}
\]

\[
\begin{align*}
A_{n+1} \times S^n &\longrightarrow A_{n+1} \\
S^n &\downarrow \\
X_{n+1} &\longrightarrow X_n \\
&\quad \searrow S^n \\
X_n/X_{n-1} &\approx \vee S^n
\end{align*}
\]
\[ X_{n/n-1} \rightarrow X_{n+1/n-1} \]

\[ 1 \rightarrow X_{n+1/n} \]
\[ h^n(X_{n+1/n}) \rightarrow h^n(X_{n+1/n-1}) \rightarrow h^n(X_{n/n-1}) \]

\[ h^{n+1}(X_{n+1/n}) \rightarrow h^{n+1}(X_{n+1/n-1}) \rightarrow h^{n+1}(X_{n/n-1}) \]
\[ h^n(X_{n+1/n}) \rightarrow h^n(X_{n+1/n-1}) \rightarrow h^n(X_{n/n-1}) \]

our choice of \( \delta' \)

\[ h^{n+1}(X_{n+1/n}) \rightarrow h^{n+1}(X_{n+1/n-1}) \rightarrow h^{n+1}(X_{n/n-1}) \]
\[
\begin{align*}
&X_{n/n-1} \xrightarrow{} X_{n+1/n-1} \\
&1 \xrightarrow{} X_{n+1/n}
\end{align*}
\]

---

\[
\begin{align*}
&h^n(X_{n+1/n}) \xrightarrow{\text{trivial}} h^n(X_{n+1/n-1}) \xrightarrow{\text{inj}} h^n(X_{n/n-1}) \\
&h^{n+1}(X_{n+1/n}) \xrightarrow{\text{surj}} h^{n+1}(X_{n+1/n-1}) \xrightarrow{\text{inj}} h^{n+1}(X_{n/n-1}) \xrightarrow{\text{trivial}} \text{coker}(\delta')
\end{align*}
\]
$h^n(X_{m+1/m}) \rightarrow h^n(X_{m+1}) \rightarrow h^n(X_m) \rightarrow h^{n+1}(X_{m+1/m})$

If $n \neq m, m+1$, both ends trivial, $h^n(X_{m+1}) \cong h^n(X_m)$

three possible values:

\[
\begin{align*}
    h^n(X_{n-1}) & \cong h^n(X_{n-2}) \cong \ldots \cong h^n(X_0), \text{ trivial} \\
    h^n(X_n) \\
    h^n(X_{n+1}) & \cong h^n(X_{n+2}) \cong \ldots \cong h^n(X)
\end{align*}
\]
$coker(\delta'_n) \cong h^n(X_{n/n-2}) \leftarrow h^n(X_{n+1/n-2}) \cong h^n(X)$

$\cong ker(\delta'_{n+1})$

eq. class inject
Chasing the diagram,

\[ h^n(X) \cong \ker(\delta'_{n+1})/\text{im}(\delta'_n) \]