new paths

univalence

loop

A

loop

loop

?
I know how to guarantee a combinatorial structure has enough paths.

My group knows how to bring that into the design of type theory.

*Guillaume Brunerie and Daniel R. Licata are also pioneers.*
1. What are the types? (form)
2. What are the constructors? (intro)
3. How to consume an element? (elim)
4. What if a constructor is consumed? ($\beta$)
5. Uniqueness principle? ($\eta$)
6. How to compose stuff? (Kan operators)
Homogeneous Compositions

*not changing type*
should work with substitution [BCH]
variant: diagonal faces and alternative filling directions
Coercion
variant 1: alternative coercion directions
variant 2: freezing parts of the input

*(used in cubical Agda)*
With these two operators every type has enough paths
They also give heterogeneous compositions.
<table>
<thead>
<tr>
<th>Major Variants</th>
<th>[CCHM+CHM]</th>
<th>[AFH+ABCFHL+CH]</th>
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<tr>
<td>algebra on $\mathbb{I}$</td>
<td>$0$, $1$, $\wedge$, $\vee$, $\sim$, De Morgan</td>
<td>$0$, $1$</td>
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<tr>
<td>homogeneous composition</td>
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<td>coercion</td>
<td>variant 2</td>
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<tr>
<td>ready-to-use proof assistants</td>
<td>cubical Agda</td>
<td>redtt</td>
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</table>
\[ A : U \]

\[ M : A \]

\[ \begin{align*}
  j=0, & \quad k:\mathbb{I} \vdash N_1 : A \quad [k=0 \mapsto M] \\
  i=1, & \quad k:\mathbb{I} \vdash N_2 : A \quad [k=0 \mapsto M, j=0 \mapsto N_1] \\
  j=1, & \quad k:\mathbb{I} \vdash N_3 : A \quad [k=0 \mapsto M, i=1 \mapsto N_2]
\end{align*} \]

\[ \begin{align*}
  \text{(faces are unordered in CCHM+CHM)}
\end{align*} \]

\[ \text{hcomp}^k A \quad [j=0 \mapsto N_1, i=1 \mapsto N_2, j=1 \mapsto N_3] \]

\[ M : A \]

\( \text{homogeneous} \)
\[ \text{comp}^k A \left[ \ldots \right] M : A[1/k] \]

\[ k : \mathbb{I} \vdash A : U \]

\[ M : A[0/k] \]
transp\textsuperscript{j} A (\sim i) M : A[1/j]

this represents (\sim i = 1) = (i = 0)
in general, r:\mathbb{I} to represent r = 1
Constraints in Contexts

\( \varphi \vdash M : A \)

\[ r := 0 \mid 1 \mid i \mid r_1 \land r_2 \mid r_1 \lor r_2 \mid \sim r \quad \text{(De Morgan)} \]

\[ \varphi := \text{false} \mid \text{true} \mid (r = 0) \mid (r = 1) \mid \varphi_1 \land \varphi_2 \mid \varphi_1 \lor \varphi_2 \]
\( r := 0 \mid 1 \mid i \mid r_1 \land r_2 \mid r_1 \lor r_2 \mid \sim r \) (De Morgan)

\( \varphi := \text{false} \mid \text{true} \mid (r = 0) \mid (r = 1) \mid \varphi_1 \land \varphi_2 \mid \varphi_1 \lor \varphi_2 \)

\( r \mapsto r=1 \) preserves \( \land, \lor, \) and \( \sim \) where \( \sim r=1 \) means \( r=0 \)

Any \( \varphi \) is equivalent to \( r=1 \) for some \( r \)

\( \text{e.g., } (i=0) \lor (i=1) = (\sim i=1) \lor (i=1) = (\sim i \lor i)=1 \)

\( \text{e.g., } \text{trapns}^j A \ r M \)
Restricted by Partial Elements

\[ M : A [\varphi \mapsto N] \]

\[ M : A \text{ and } \varphi \vdash M \equiv N : A \]
2. What are the constructors? (intro)
3. How to consume an element? (elim)
4. What if a constructor is consumed? (β)
5. Uniqueness principle? (η)

6. hcomp and transp (and thus comp)

✓ Gives us all the paths
✓ Definable for every type
\[
\begin{align*}
M : T & \quad \phi, i : \exists \vdash N : T \\
\text{hcomp}^i T [\phi \mapsto N] M & \equiv M : T
\end{align*}
\]

\[
\begin{align*}
M : T & \quad r : \exists \\
\text{transp}^i T r M & \equiv M : T
\end{align*}
\]
new
paths

univalence
loop

hcomp
transp

✓ the unit
the empty type
functions
pairs
paths
the circle
universes

(many others)