Homework 4: W and PoP

Due 2019/04/03 (Fri) Anywhere on Earth

We are going to meet a very general inductive type $\mathcal{W}(A; x.B)$ that can be used to implement a wide range of other inductive types. We will also practice on paths over paths. You should send updated `hw4-handout.agda` to Favonia when you are done.

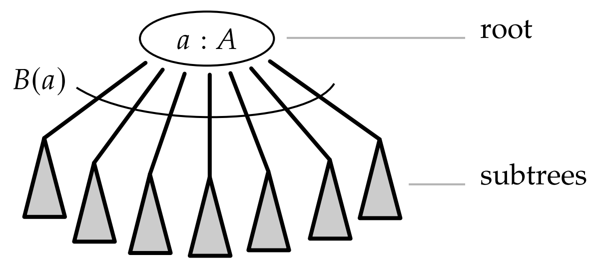
$\mathcal{W}$-Types

The letter "$\mathcal{W}$" stands for *wellordering*, which is closely related to *(transfinite) induction.* $\mathcal{W}$-types intend to capture induction itself, and as a result, they become one of the most general inductive types; a wide range of inductive types can be implemented using $\mathcal{W}$-types. That said, it does not handle newcomers such as the circle in the homotopy type theory.

An element in a $\mathcal{W}$-type can be visualized as a well-founded tree. A $\mathcal{W}$-type is indexed by its branching signature, which consists of a type A and a family of types B indexed by A:

- The type A represents all variations of nodes. For example, if A is the Boolean type $\mathbb{2}$, it means there are exactly two kinds of nodes in trees. We will see how we need exactly two kinds of nodes to implement natural numbers.
- For a tree node of kind $a : A$, the type $B(a)$ indexes all its subtrees. If $B(a)$ is a finite type of three elements, it means a tree node of kind a has exactly three branches. If $B(a)$ is the empty type, it means there is no subtree for any node of kind a. The index type $B(a)$ can vary with a and does not need to be finite—for example, it can be $\mathbb{N}$.

Here is a diagram showing the visualization:



We write $\mathcal{W}(A; x.B)$ to make explicit its branching signature.

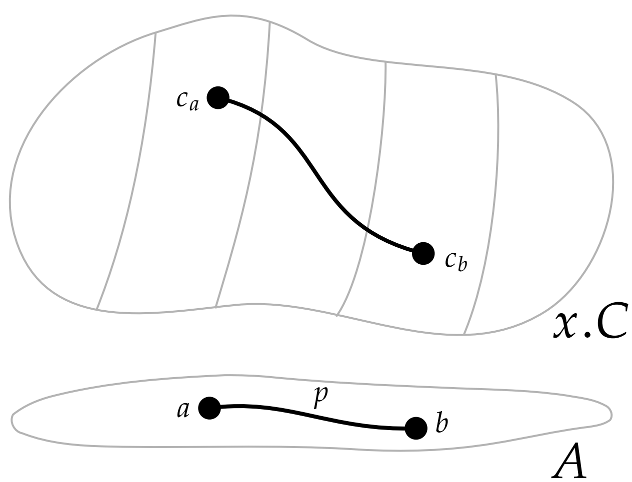
Formally, there is only one constructor, `sup`, in the type $\mathcal{W}(A; x.B)$. This constructor takes an element $a : A$ as the kind of the root, and a function of type $B[a/x] \to \mathcal{W}(A; x.B)$ representing the collection of subtrees indexed by $B[a/x]$. The constructor is called `sup` because it is similar to taking the supremum of a collection of ordinals to form a larger ordinal; here, we are taking the "supremum" of a collection of subtrees to form a larger tree.

A wide range of inductive types can be implemented using $\mathcal{W}$-types and the basic types we had before introducing the universes (functions, pairs, the unit, the empty type, and disjoint sums) with the help of function extensionality. Please figure out a way to implement the natural numbers and reflect upon the usage of function extensionality.

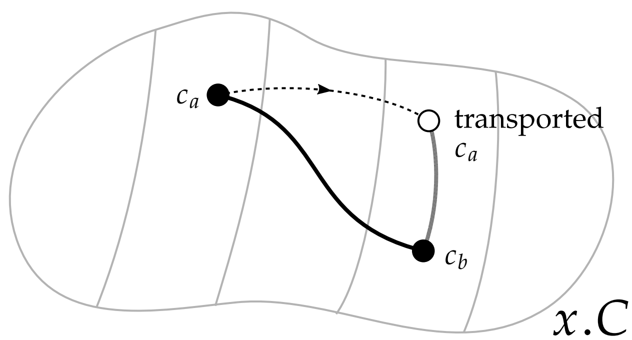
Task 1. *Finish Part I of the Agda file.* **(Hint)** *Hints are in the file.*

Paths over Paths

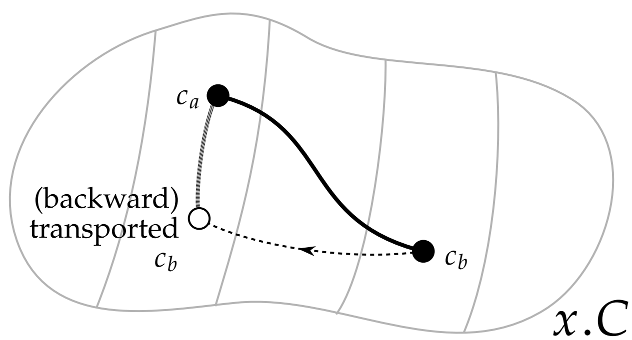
Paths over paths will play an even more signifiant role in cubical type theory and are thus worth studying. Here is the cloud view of them:



The first interesting fact is that there are actually three ways to define paths over paths. Pictorially, what Favonia showed in lectures was a direct definition by pattern matching on the groud path p. We can also break it down into a horizontal path across fibers in the cloud and a vertical one within a fiber. More precisely, we can horizontally move one of the endpoints to the fiber where the other endpoint belong, and then vertically identify the two points now in the same fiber. The horizontal movement is determined by the definition of the cloud ($x.C$) and the groud path p, and we only specify the vertical movement within the fiber. Suppose we first move c_a to the fiber over b. The picture looks like this:



Symmetrically, one can also move c_b back to the fiber where c_a belongs:



The horizontal movement induced by $x.C$ is called *transport*: as long as there is a path p from a to b on the ground, we can always transport elements between the fibers $C[a/x]$ and $C[b/x]$. This should not be surprising, because the equality nature of p means that a and b are somewhat "equal" and thus any proof of $C[a/x]$ should lead to a proof of $C[b/x]$ and vice versa. The substitutional point of view gives the transportation another name: *substitution*, which you would see in the literature. What is new here is that the cloud view promoted by the homotopy-theoretic interpretation, and this view shows that the higher-dimensional structure in $x.C$ could lead to interesting transportations in the sky. Indeed, we will see lots of twisted clouds when moving onto cubical type theory.

Task 2. *Finish Part II.1. Please give two different definitions of paths over paths using the above idea, and prove one of them is equivalent to the definition given in lectures.*

Bonus Task 1. *Prove* both *definitions are equivalent to the one given in lectures.*

Task 3. *Finish Part II.2. Please finish the definition of basic operations on paths over paths: path concatenation and path reversal.*

Grading: As usual, only one letter grade (without plus or minus) will be assigned to the *entire* homework according to the criterion explained in the syllabus. Bonus tasks are purely for your enjoyment and will not affect your grading in any way.