Homework 1: Heyting Algebra

Due 2019/02/14 (Fri) Anywhere on Earth

The goal of this homework is to practice writing derivations and understand the algebraic (order-theoretic) viewpoint of provability in type theory. In particular, we will focus on the *Lindenbaum–Tarski algebra*, named after the fantastic logicians Adolf Lindenbaum and Alfred Tarski. The essential idea is that if we can prove B under the assumption A, then we say the type A is *less than* the type B. (The less, the more powerful.) We then group logically equivalent types as equivalent classes to make the relation a partial order, because a partial order needs to be antisymmetric. We can then analyze provability via this partial order through the lens of order theory.

1 Formal Definitions

Given a predicate logic theory, one can define the relation $\leq$ among formulas as follows:

$$A \leq B \text{ if and only if } A \vdash B.$$

Two formulas A and B are logically equivalent if $A \leq B$ and $B \leq A$. Consider the formulas quotiented by the logical equivalence. The equivalence classes and the partial order $\leq$ form an interesting algebra, and it is called the Lindenbaum–Tarski algebra of this logic theory, but . . . this is a course about type theories, not logic theories!

We can mimic the construction for a (non-dependent) *type theory*[1] by defining the relation $\leq$ among (closed) types as follows:

$$A \leq B \text{ if and only if there is } x.M \text{ such that } x{:}A \vdash M : B \text{ is derivable.}$$

As expected, two types A and B are logically equivalent if $A \leq B$ and $B \leq A$. To make $\leq$ a partial order, we quotient the types by logical equivalence.

[1] For example, the type theory we discussed up to the 2020/01/30 lecture.

The same algebra can also be constructed from the category in the 2020/01/30 lecture by defining $\leq$ as follows:

$$A \leq B \text{ if and only if } \mathsf{Mor}(A, B) \text{ is not empty.}$$

2 Heyting Algebra

The Lindenbaum–Tarski algebra of the type theory consisting of $\times$, $\top$, $+$, $\bot$ and $\to$ is a bounded lattice:

- The unit type $\top$ is the top (greatest element) $\mathbb{1}$ because for any (closed) type A,
$$\frac{}{x{:}A \vdash \diamond : \top}$$
- The empty type $\bot$ is the bottom (least element) $\mathbb{0}$ because for any (closed) type A,
$$\frac{}{x{:}\bot \vdash \texttt{abort}(x) : A}$$
- $A \times B$ is the meet $(A \wedge B)$ of A and B. (I will prove this.)
- $A + B$ is the join $(A \vee B)$ of A and B. (*You* will prove this.)

`01/31 07:00pm` The order-theoretic notations were added for clarifying the bonus tasks on the last page.

Moreover, it is a *Heyting algebra*! A Heyting algebra is a bounded lattice equipped with a binary operator $\supset$ where $A \supset B$ is the greatest element such that $A \times (A \supset B) \leq B$. Unsurprisingly, $A \supset B$ can be implemented as the function type $A \to B$.

Remark 1. *In fact, the algebra we constructed out of this type theory is the most general Heyting algebra, in the sense that there is a unique homomorphism to any other Heyting algebra. We will not prove this theorem in this homework.*

Remark 2. *Technically speaking, one needs to show the logical equivalence is a* congruence. *For example, if A and A' are logically equivalent, and B and B' are logically equivalent, then $A \supset B$ and $A' \supset B'$ are logically equivalent, and similarly for other operators, too. For this homework, you may take congruence for granted.*

3 Warm-up: Distributivity

Lemma 3.1. *We have* $A \times B + A \times C \leq A \times (B + C)$.

Proof: It is sufficient to show that this judgement is derivable:

$$x{:}(A{\times}B{+}A{\times}C) \vdash \texttt{case}(y.\langle \pi_1(y), \texttt{inl}(\pi_2(y))\rangle; z.\langle \pi_1(z), \texttt{inr}(\pi_2(z))\rangle; x) : A{\times}(B{+}C)$$

By the +-elim rule, it is sufficient to show the derivability of the following three judgments:

$$x{:}(A \times B + A \times C), y{:}A \times B \vdash \langle \pi_1(y), \texttt{inl}(\pi_2(y))\rangle : A \times (B + C) \tag{1}$$
$$x{:}(A \times B + A \times C), z{:}A \times C \vdash \langle \pi_1(z), \texttt{inr}(\pi_2(z))\rangle : A \times (B + C) \tag{2}$$
$$x{:}(A \times B + A \times C) \vdash x : A \times B + A \times C \tag{3}$$

Judgment (1) is derivable because of the ×-intro rule and the following two derivations:

$$\dfrac{\dfrac{}{x{:}(A \times B + A \times C), y{:}A \times B \vdash y : A \times B}\ \text{variable}}{x{:}(A \times B + A \times C), y{:}A \times B \vdash \pi_1(y) : A}\ \times\text{-elim}_1$$

$$\dfrac{\dfrac{\dfrac{}{x{:}(A \times B + A \times C), y{:}A \times B \vdash y : A \times B}\ \text{variable}}{x{:}(A \times B + A \times C), y{:}A \times B \vdash \pi_2(y) : B}\ \times\text{-elim}_2}{x{:}(A \times B + A \times C), y{:}A \times B \vdash \texttt{inl}(\pi_2(y)) : B + C}\ +\text{-intro}_{\text{left}}$$

Similarly, Judgment (2) is derivable because of the ×-intro rule and the following two derivations:

$$\dfrac{\dfrac{}{x{:}(A \times B + A \times C), z{:}A \times C \vdash z : A \times C}\ \text{variable}}{x{:}(A \times B + A \times C), z{:}A \times C \vdash \pi_1(z) : A}\ \times\text{-elim}_1$$

$$\dfrac{\dfrac{\dfrac{}{x{:}(A \times B + A \times C), z{:}A \times B \vdash z : A \times B}\ \text{variable}}{x{:}(A \times B + A \times C), z{:}A \times B \vdash \pi_2(z) : B}\ \times\text{-elim}_2}{x{:}(A \times B + A \times C), z{:}A \times C \vdash \texttt{inr}(\pi_2(y)) : B + C}\ +\text{-intro}_{\text{right}}$$

Finally, Judgment (3) is derivable:

$$\dfrac{}{x{:}(A \times B + A \times C) \vdash x : A \times B + C}\ \text{variable}$$

□

Remark 3. *We showed many subderivations instead of one single derivation because of the difficulty in typesetting.*

Task 1. *Show that $A \times (B + C) \leq A \times B + A \times C$, the other direction of distributivity.*

Corollary 1. $A \times (B + C)$ *and* $A \times B + A \times C$ *are logically equivalent.*

4 Checking Heyting-ness

Let's check carefully how the algebra is a Heyting algebra.

Lemma 4.1. $A \times B$ *is the meet of* A *and* B*.*

Proof: $A \times B \leq A$ because $x{:}A \times B \vdash \pi_1(x) : A$. Similarly, $A \times B \leq B$ because $x{:}A \times B \vdash \pi_2(x) : B$. Moreover, for any C such that $C \leq A$ and $C \leq B$, by the definition of $\leq$ we know there exist $x.M$ and $y.N$ such that $x{:}C \vdash M : A$ and $y{:}C \vdash N : B$ are derivable. By variable renaming,[2] $z{:}C \vdash M[z/x] : A$ and $z{:}C \vdash N[z/x] : B$ are derivable, as well. Therefore, we have a derivation of $z{:}C \vdash \langle M[z/x], N[z/y] \rangle : A \times B$ as follows:

$$\dfrac{\begin{array}{c}\mathcal{D}_1 \\ z{:}C \vdash M[z/x] : A\end{array} \qquad \begin{array}{c}\mathcal{D}_2 \\ z{:}C \vdash N[z/y] : B\end{array}}{z{:}C \vdash \langle M[z/x], N[z/y] \rangle : A \times B}\ \times\text{-intro}$$

which means $z{:}C \vdash \langle M[z/x], N[z/y] \rangle : A \times B$ is derivable. By definition, this implies $C \leq A \times B$. Therefore, $A \times B$ is the greatest lower bound of A and B. □

Task 2. *Prove that $A + B$ is the join of A and B.* **(Hint)** *Pay close attention to the contexts in the rules! You might want to use some of these (existing or admissible) structural rules: weakening, contraction, exchange, substitution, renaming of bound variables, ...*

Here is the universal property of the complement $\neg A$, which is defined to be $A \to \bot$. This is a special case of the more general theorem that $A \to B$ satisfies the requirements of $A \supset B$. (You may prove the more general theorem directly.)

Task 3. *Prove that $\neg A$ is the greatest element inconsistent with A (i.e., $A \times \neg A \leq \bot$).*

[2]We assume every judgement respects α-equivalence (renaming of bound variables), which should be the case in any reasonable type theory. Otherwise, variable renaming can be done by the weakening and substitution rules.

5 More Order Theory

Distributivity actually works in any Heyting algebra! This is related to the fact that the lattice we were working in is the most general Heyting algebra.

Bonus Task 1. *Prove that distributivity holds in* any *Heyting algebra, not just the algebra induced by the type theory. That is,*

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C).$$

(Hint) *The operator* $\supset$ *can help.*

More interesting facts from order theory:

Bonus Task 2. *Show that* $(A \supset \mathbb{0}) \vee B$ *satisfies the universal property of* $A \supset B$ *in any Boolean algebra, which can be defined as a Heyting algebra satisfying* $\mathbb{1} \leq A \vee (A \supset \mathbb{0})$ *(the order-theoretic LEM). You should rely on universal properties instead of models based on truth tables.*

`01/31 07:00pm` The bonus tasks were rewritten using the correct notations.

6 Grading and Formatting

Only one letter grade (without plus or minus) will be assigned to the *entire* homework according to the criterion explained in the syllabus. Bonus tasks will *not* (positively or negatively) affect your grades.

The source of this document is only for your reference. You do not have to follow the style or keep the text. It is more important that Favonia can understand which task you are working in.

7 LaTeX Tips

Read the documentations of the packages `ebproof` (recommended for derivation trees) and `mathpartir` (also working).