Solution to Homework 1: Heyting Algebra

Task 1. *Show that $A \times (B + C) \leq A \times B + A \times C$, the other direction of distributivity.*

Solution: It is sufficient to show that this judgement is derivable:

$x{:}(A\times(B+C)) \vdash \texttt{case}(y.\texttt{inl}(\langle \pi_1(x), y\rangle); z.\texttt{inr}(\langle \pi_1(x), z\rangle); \pi_2(x)) : A\times B+A\times C$

By the +-elim rule, it is sufficient to show the derivability of the following three judgments:

$$x{:}(A \times (B + C)), y{:}B \vdash \texttt{inl}(\langle \pi_1(x), y\rangle) : A \times B + A \times C \tag{1}$$
$$x{:}(A \times (B + C)), z{:}C \vdash \texttt{inr}(\langle \pi_1(x), z\rangle) : A \times B + A \times C \tag{2}$$
$$x{:}(A \times (B + C)) \vdash \pi_2(x) : B + C \tag{3}$$

Judgment (1) is derivable:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x{:}(A \times (B + C)), y{:}B \vdash x : A \times (B + C)}\ \text{variable}}{x{:}(A \times (B + C)), y{:}B \vdash \pi_1(x) : A}\ \times\text{-elim}_1 \qquad \dfrac{}{x{:}(A \times (B + C)), y{:}B \vdash y : B}\ \text{variable}}{x{:}(A \times (B + C)), y{:}B \vdash \langle \pi_1(x), y\rangle : A \times B}\ \times\text{-intro}}{x{:}(A \times (B + C)), y{:}B \vdash \texttt{inl}(\langle \pi_1(x), y\rangle) : A \times B + A \times C}\ +\text{-intro}_{\text{left}}$$

Similarly, Judgment (2) is derivable:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x{:}(A \times (B + C)), z{:}C \vdash x : A \times (B + C)}\ \text{variable}}{x{:}(A \times (B + C)), z{:}C \vdash \pi_1(x) : A}\ \times\text{-elim}_1 \qquad \dfrac{}{x{:}(A \times (B + C)), z{:}C \vdash z : C}\ \text{variable}}{x{:}(A \times (B + C)), z{:}C \vdash \langle \pi_1(x), z\rangle : A \times C}\ \times\text{-intro}}{x{:}(A \times (B + C)), z{:}C \vdash \texttt{inr}(\langle \pi_1(x), z\rangle) : A \times B + A \times C}\ +\text{-intro}_{\text{right}}$$

Finally, Judgment (3) is derivable:

$$\dfrac{\dfrac{}{x{:}(A \times (B + C)), z{:}C \vdash x : A \times (B + C)}\ \text{variable}}{x{:}(A \times (B + C)) \vdash \pi_2(x) : B + C}\ \times\text{-elim}_2$$

Task 2. *Prove that $A + B$ is the join of A and B.* **(Hint)** *Pay close attention to the contexts in the rules! You might want to use some of these (existing or admissible) structural rules: weakening, contraction, exchange, substitution, renaming of bound variables, ...*

Solution: $A \le A + B$ because $x{:}A \vdash \mathtt{inl}(x) : A + B$. Similarly, $B \le A + B$ because $x{:}B \vdash \mathtt{inr}(x) : A + B$. Moreover, for any C such that $A \le C$ and $B \le C$, by the definition of $\le$ we know there exist $x.M$ and $y.N$ such that $x{:}A \vdash M : C$ and $y{:}B \vdash N : C$ are derivable. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the derivations, respectively. The judgment $z{:}A + B \vdash \mathtt{case}(x.M; y.N; z) : C$ can be derived as follows:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ x{:}A \vdash M : C\end{matrix}}{x{:}A, z{:}A + B \vdash M : C}\,\text{weakening}}{z{:}A + B, x{:}A \vdash M : C}\,\text{exchange} \qquad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_2\\ y{:}B \vdash N : C\end{matrix}}{y{:}B, z{:}A + B \vdash N : C}\,\text{weakening}}{z{:}A + B, y{:}B \vdash N : C}\,\text{exchange} \qquad \dfrac{}{z{:}A + B \vdash z : A + B}\,\text{variable}}{z{:}A + B \vdash \mathtt{case}(x.M; y.N; z) : C}\,\text{+-elim}$$

By definition, this implies $A + B \le C$. Therefore, $A + B$ is the least upper bound of A and B.

Bonus Task 1. *Prove that $A \to B$ satisfies the universal property of $A \supset B$.*

Solution: $A \times (A \to B) \le B$ because

$$\dfrac{\dfrac{\dfrac{}{x{:}A \times (A \to B) \vdash x : A \times (A \to B)}\,\text{variable}}{x{:}A \times (A \to B) \vdash \pi_2(x) : A \to B}\,\times\text{-elim}_2 \qquad \dfrac{\dfrac{}{x{:}A \times (A \to B) \vdash x : A \times (A \to B)}\,\text{variable}}{x{:}A \times (A \to B) \vdash \pi_2(x) : A}\,\times\text{-elim}_1}{x{:}A \times (A \to B) \vdash \pi_2(x)(\pi_1(x)) : B}\,\to\text{-elim}$$

Suppose we have a type C such that $A \times C \le B$. By definition of $\le$, it means there is $x.M$ such that there is a derivation $\mathcal{D}$ of the judgment $x{:}A \times C \vdash M : B$. We know $C \le (A \to B)$ because the judgment

$$z{:}C \vdash \lambda(y{:}A).M[\langle y, z\rangle/x] : A \to B$$

is derivable:

$$
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  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{\mathcal{D}}{x{:}A \times C \vdash M : B}
        }{x{:}A \times C, z{:}C \vdash M : B}\ \text{weakening}
      }{x{:}A \times C, z{:}C, y{:}A \vdash M : B}\ \text{weakening}
    }{z{:}C, x{:}A \times C, y{:}A \vdash M : B}\ \text{exchange}
  }{z{:}C, y{:}A, x{:}A \times C \vdash M : B}\ \text{exchange}
  \qquad
  \dfrac{
    \dfrac{}{z{:}C, y{:}A \vdash y : A}\ \text{variable}
    \qquad
    \dfrac{}{z{:}C, y{:}A \vdash z : C}\ \text{variable}
  }{z{:}C, y{:}A \vdash \langle y, z \rangle : A \times C}\ \times\text{-intro}
}{
  \dfrac{z{:}C, y{:}A \vdash M[\langle y, z \rangle / x] : B}{z{:}C \vdash \lambda(y{:}A).M[\langle y, z \rangle / x] : A \to B}\ \to\text{-intro}
}\ \text{substitution}
$$

Therefore, $A \to B$ is the greatest element such that $A \times (A \to B) \leq B$.