Covering Spaces in Homotopy Type Theory

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This material is based upon work supported by the National Science Foundation under Grant No. 1116703.

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Why bother?

Fundamental Groups!

[computer checked]

This work is covered by Agda

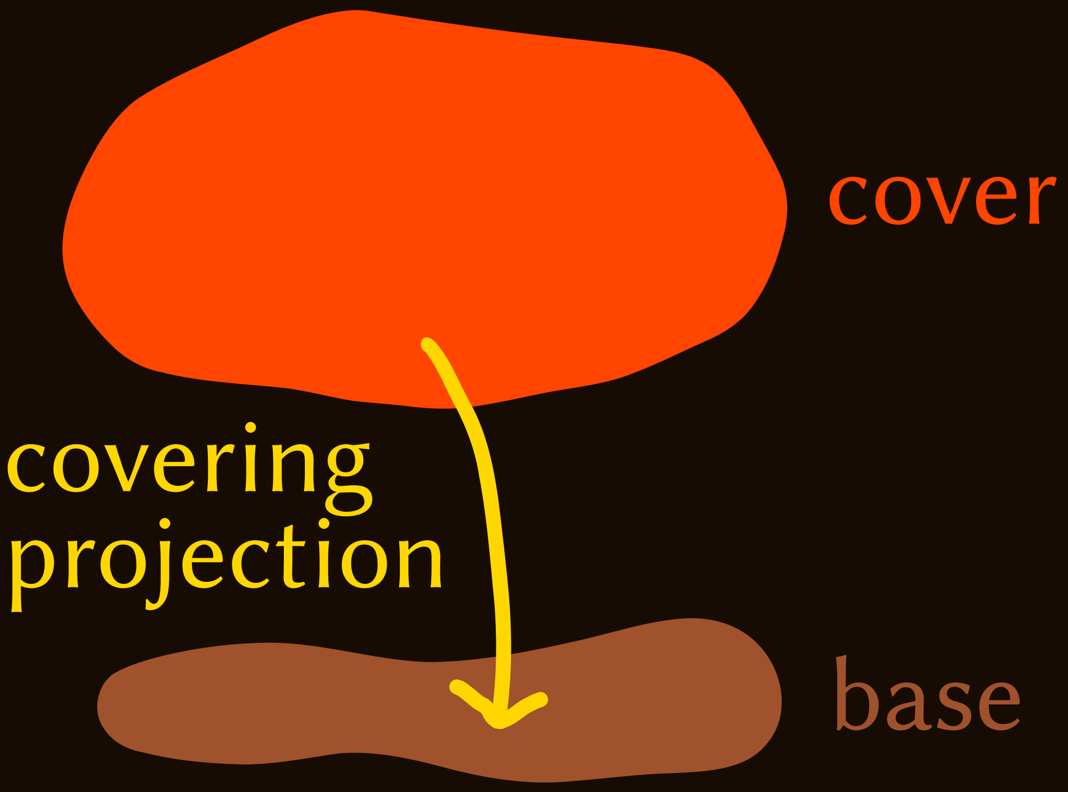
Covered Topics

Classification
Universality

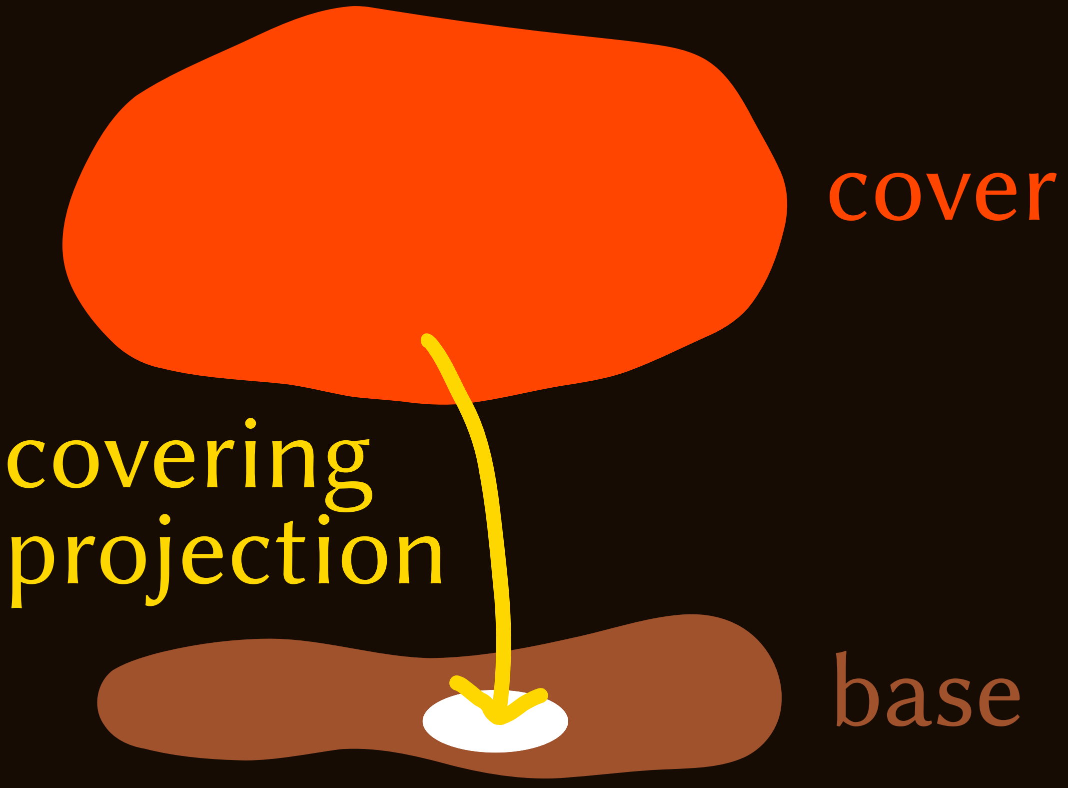
Part 0

Definition

Definition of Covering Spaces



Definition of Covering Spaces



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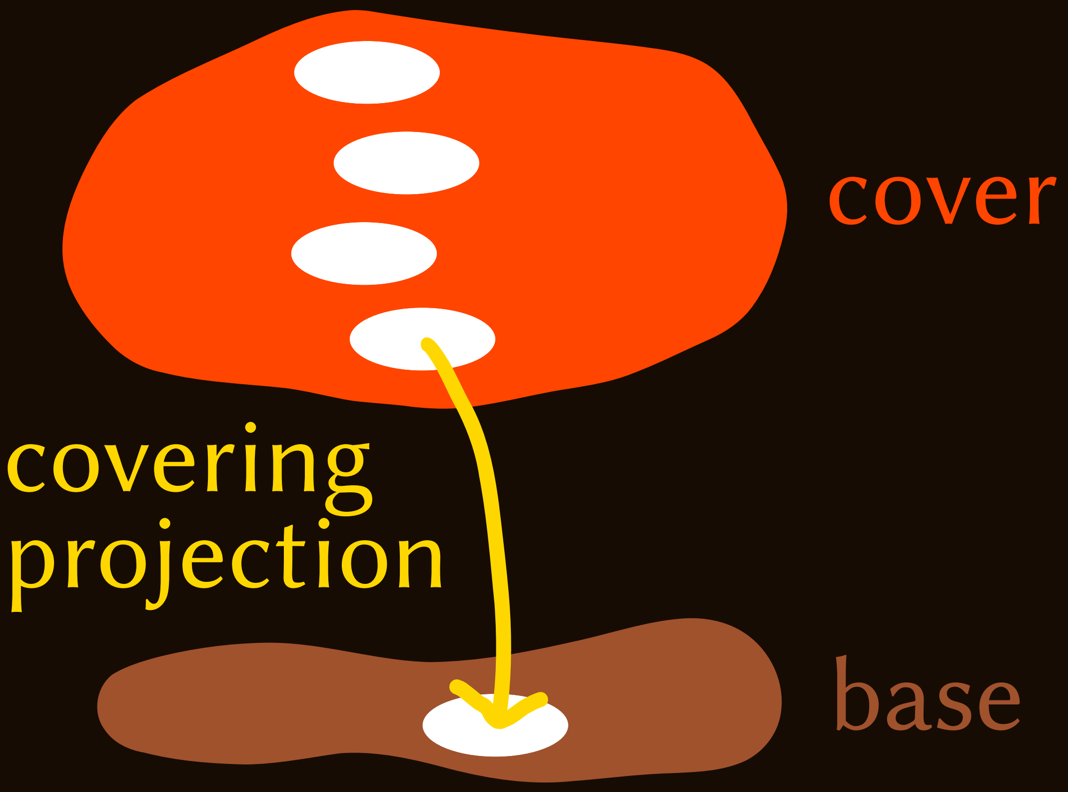

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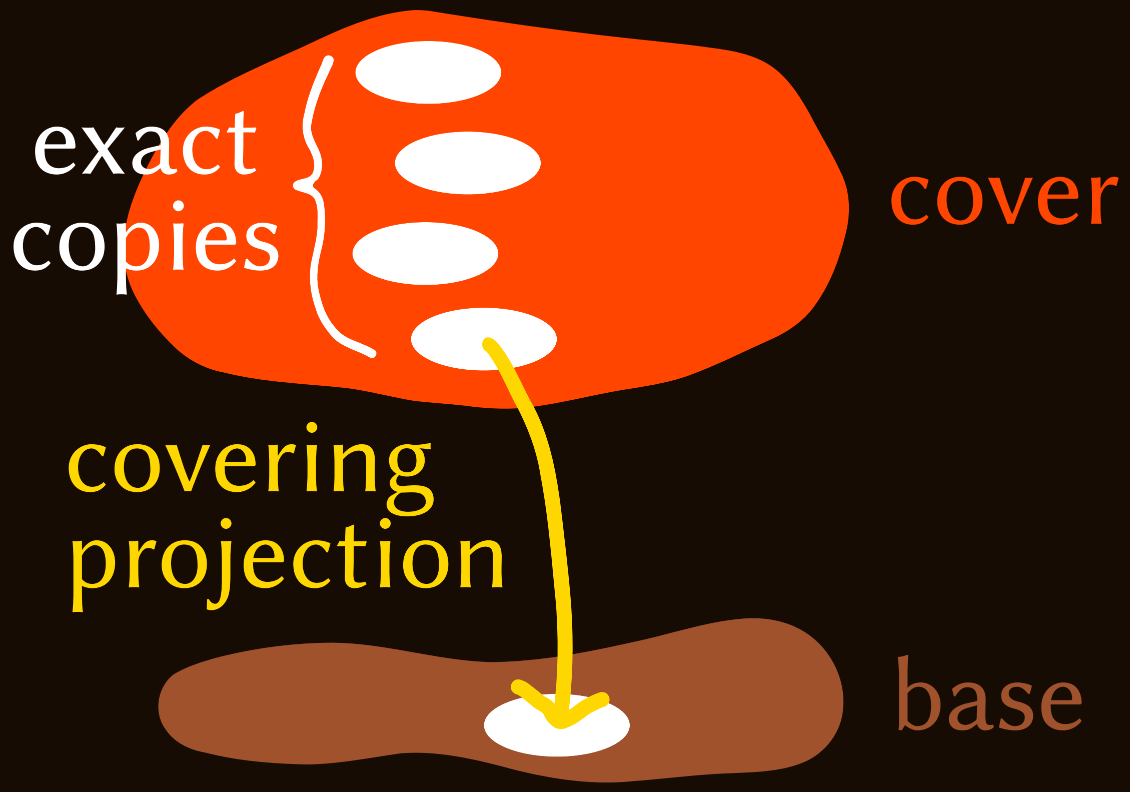

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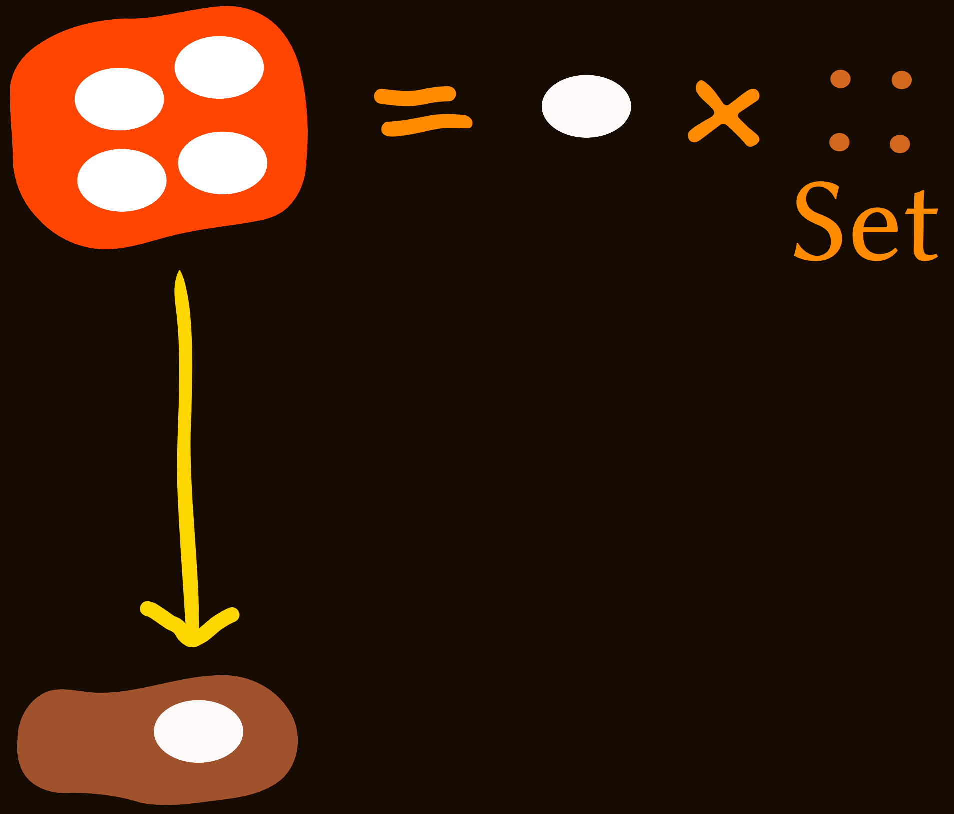
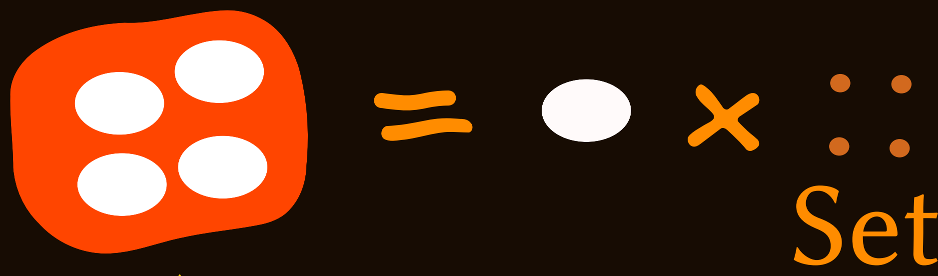
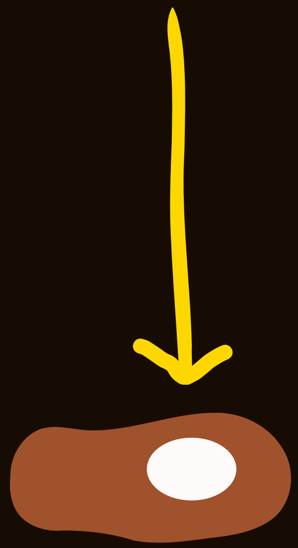


Definition of Covering Spaces

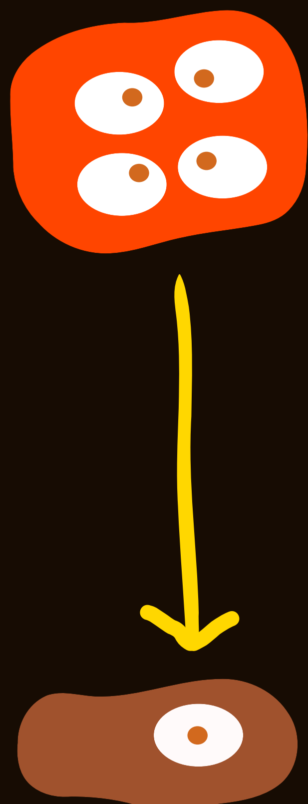


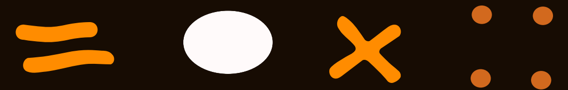


HoTT
True Facts #28

Continuity
is free!

Definition of Covering Spaces





Set

HoTT
True Facts #28

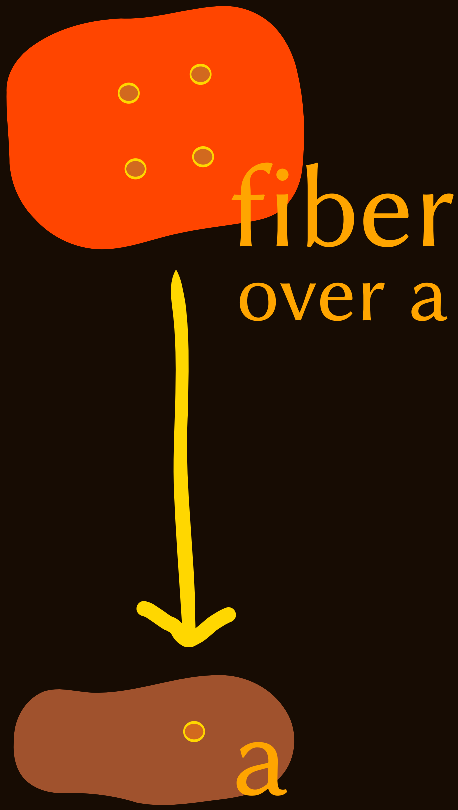
Continuity
is free!

Definition of Covering Spaces

$$A \longrightarrow \mathbf{Set}$$

Cover over A

Definition of Covering Spaces



$$A \longrightarrow \mathbf{Set}$$

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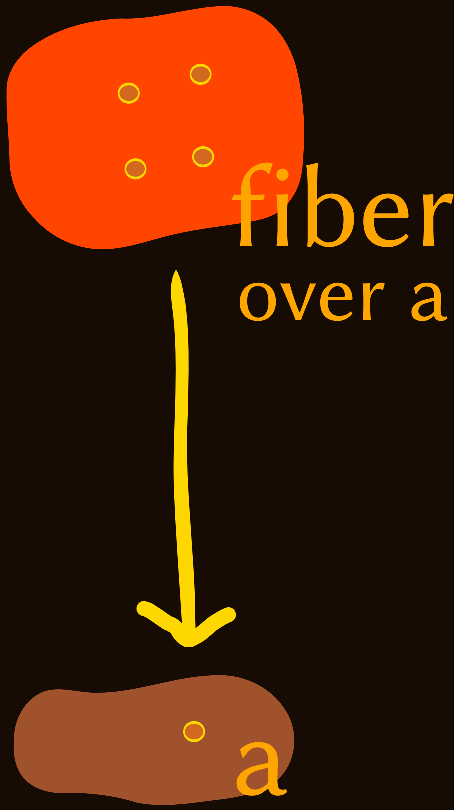
Definition of Covering Spaces



$$A \longrightarrow \mathbf{Set}$$

Cover over A

It is a functor!

Definition of Covering Spaces

$a_1 \rightarrow set_1$

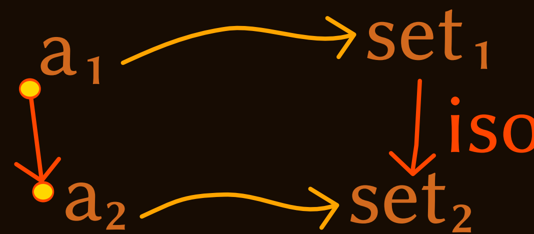
$a_2 \rightarrow set_2$

$$A \longrightarrow Set$$

Cover over A

It is a functor!

Definition of Covering Spaces



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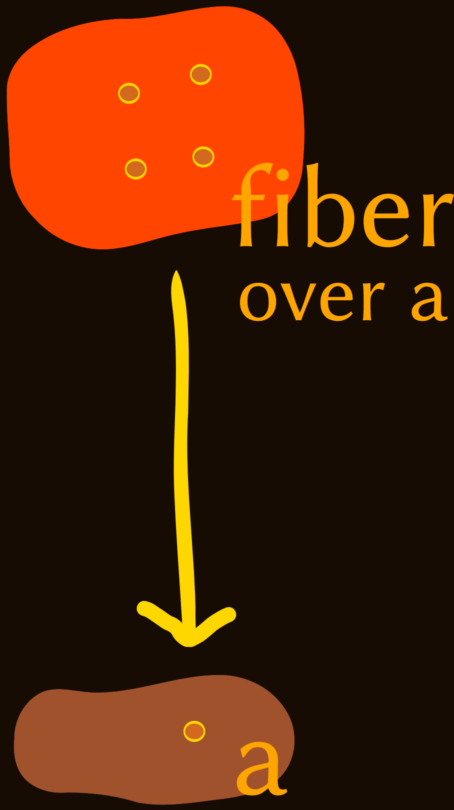
Definition of Covering Spaces



$$A \longrightarrow \mathrm{Set}$$

Cover over A

path-connected?

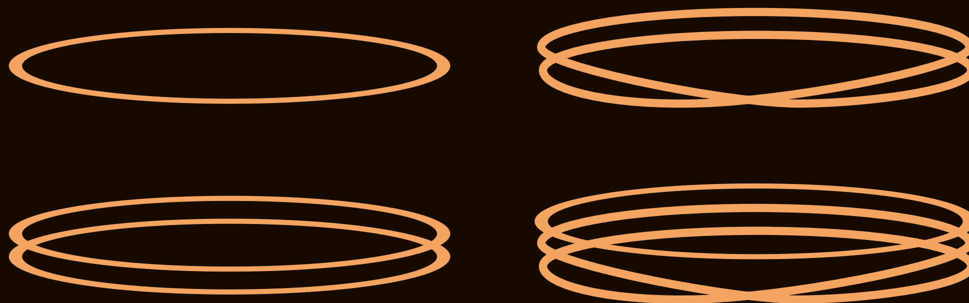
Definition of Covering Spaces

$$A \longrightarrow \mathbf{Set}$$

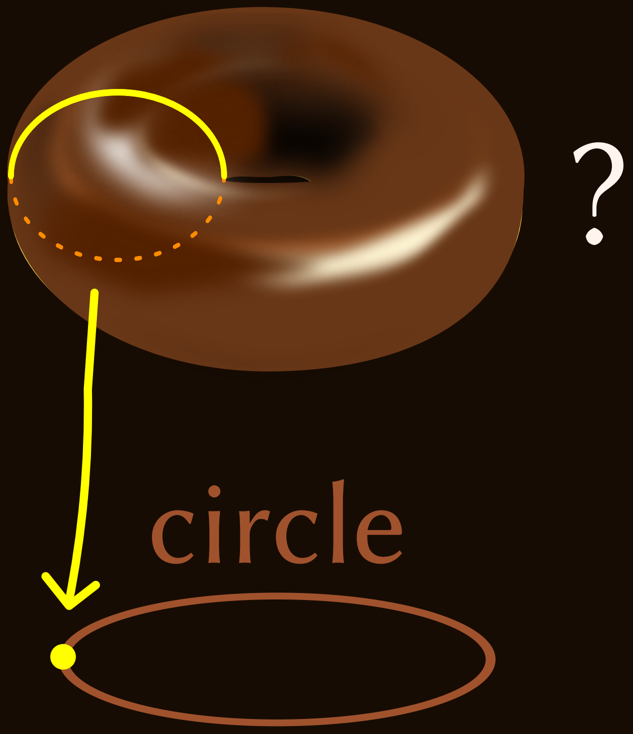
Cover over A

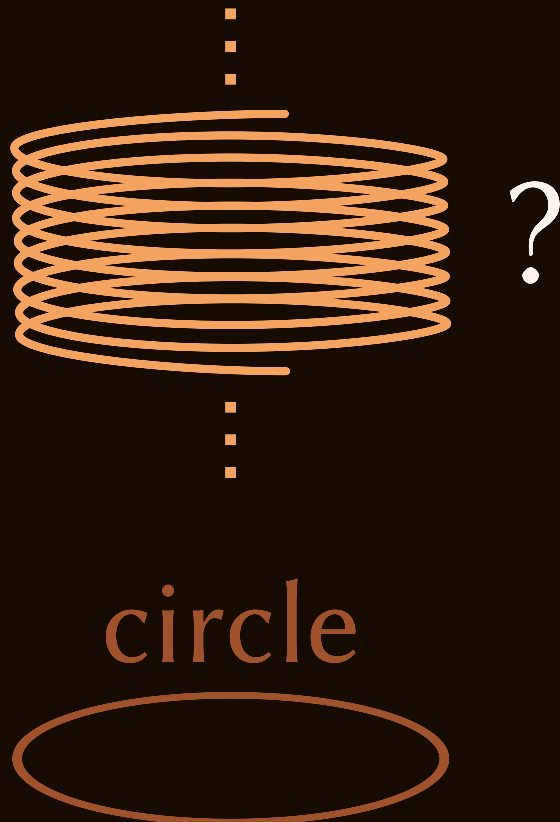
path-connected?

pointed?



circle


?
circle

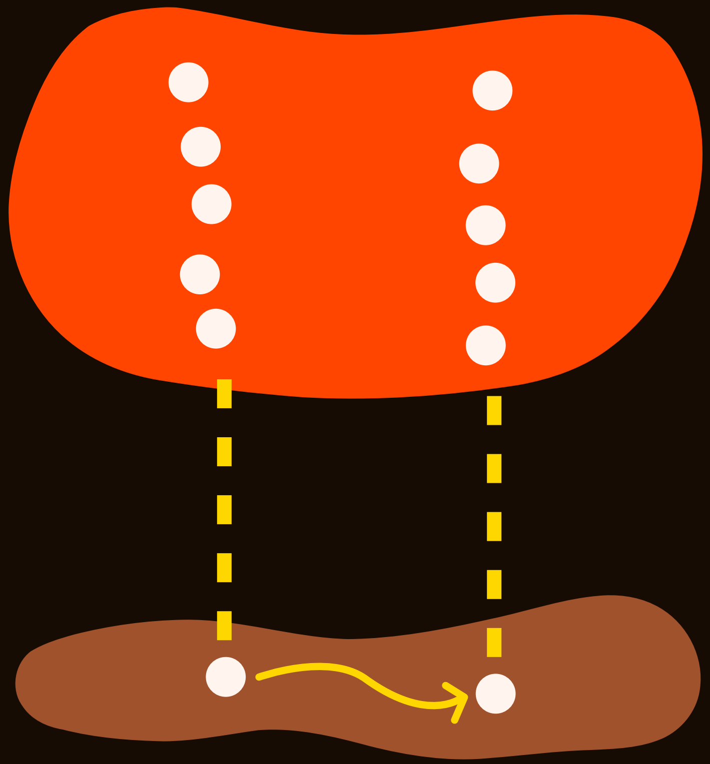

?
circle

Part 1

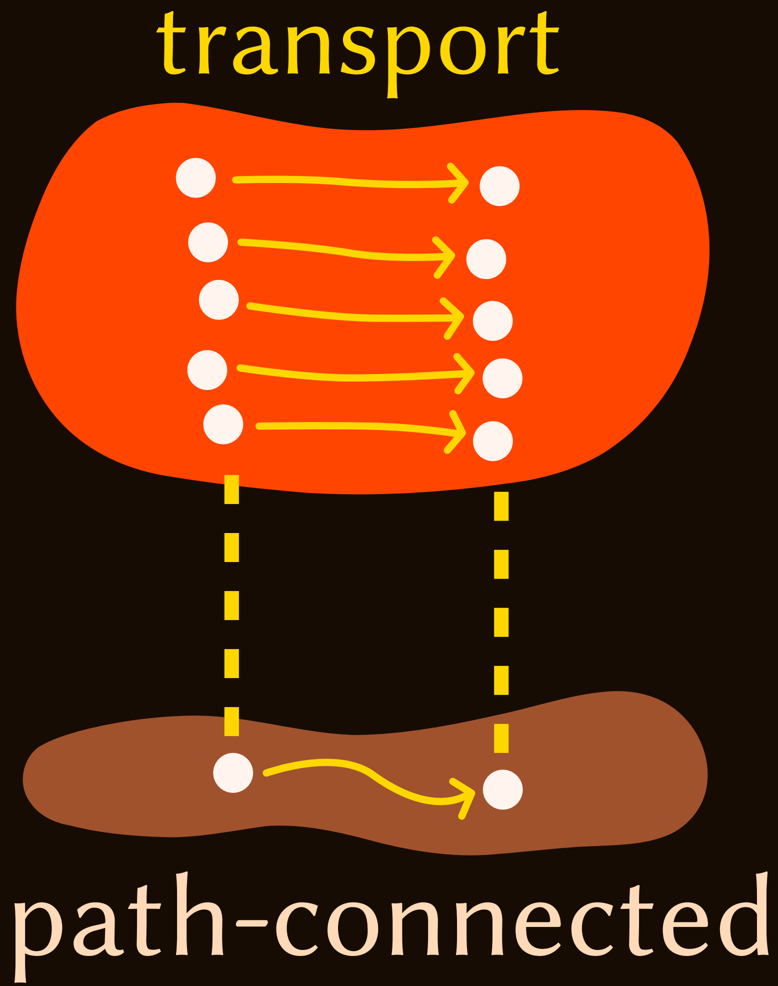
Classification

Goal

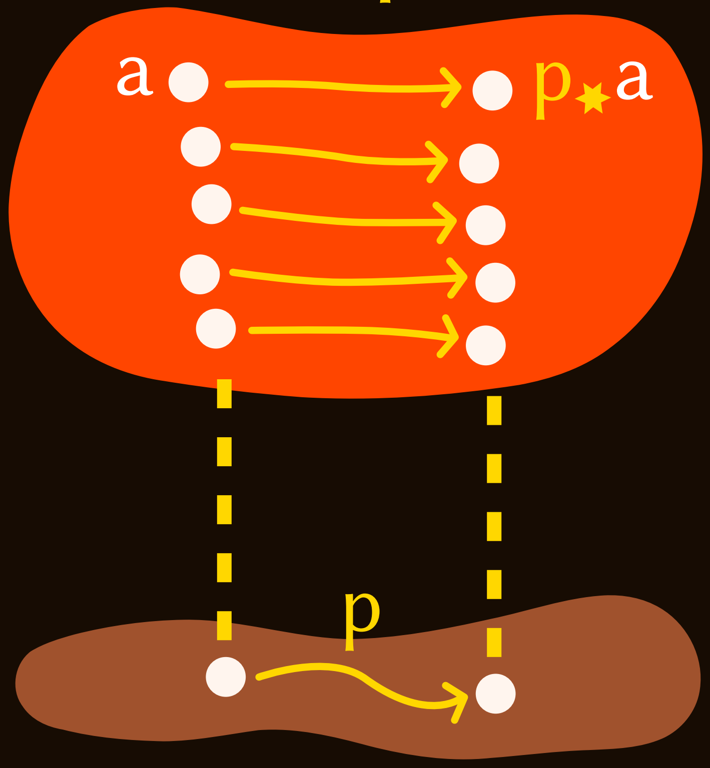
Find representations of covering spaces



path-connected


transport
path-connected

transport

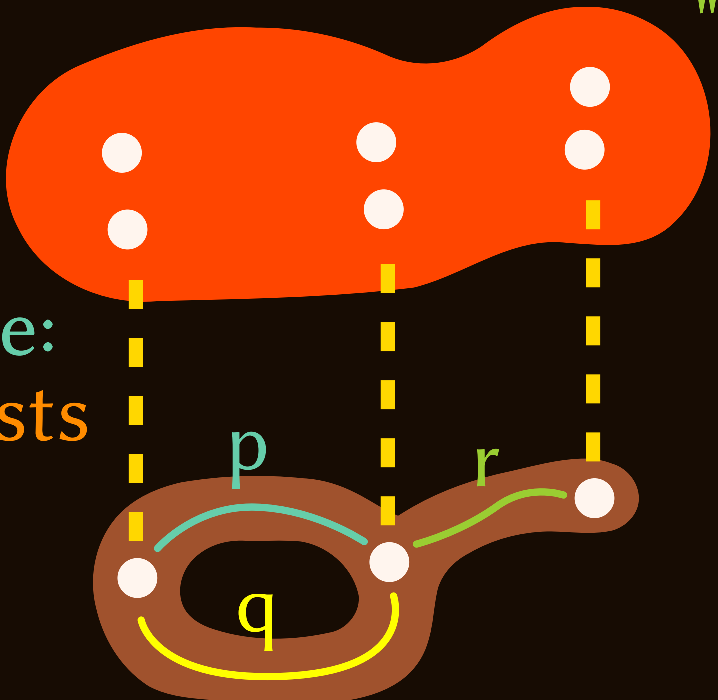


path-connected

For example...

Green part:
"fixable"

Yellow + Blue:
inherent twists



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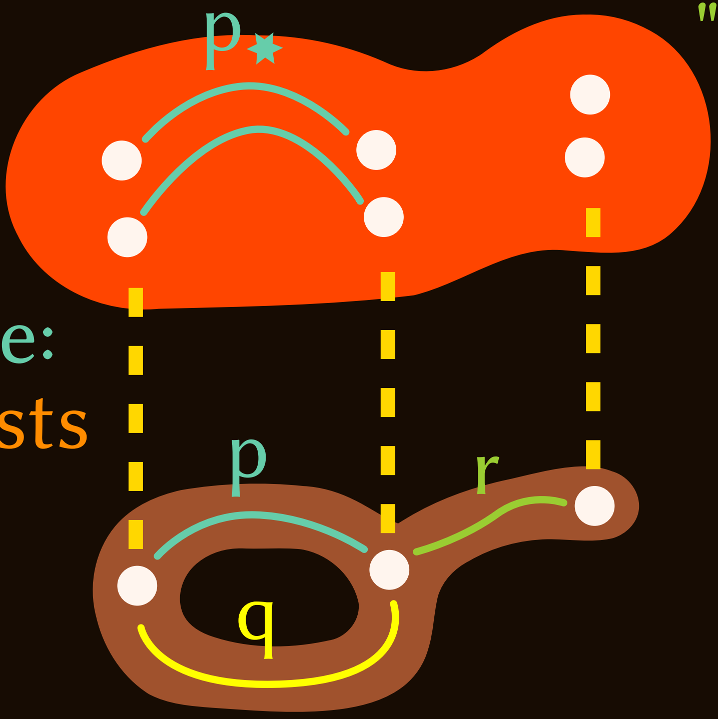

For example...

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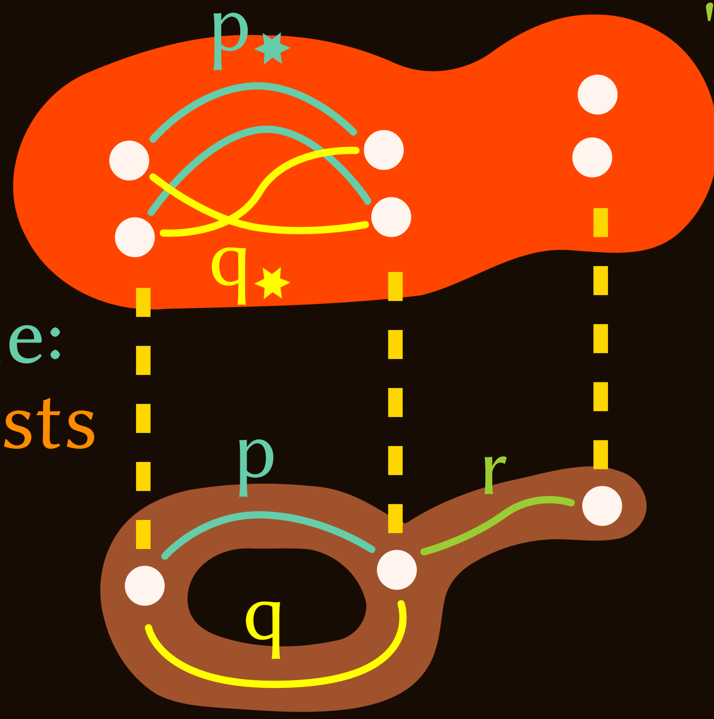
"fixable"



Yellow + Blue:
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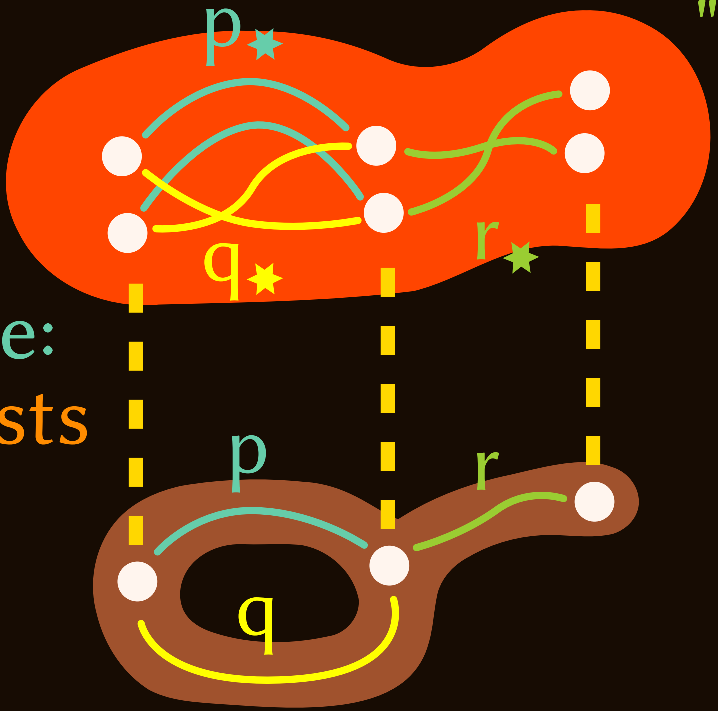

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For example...

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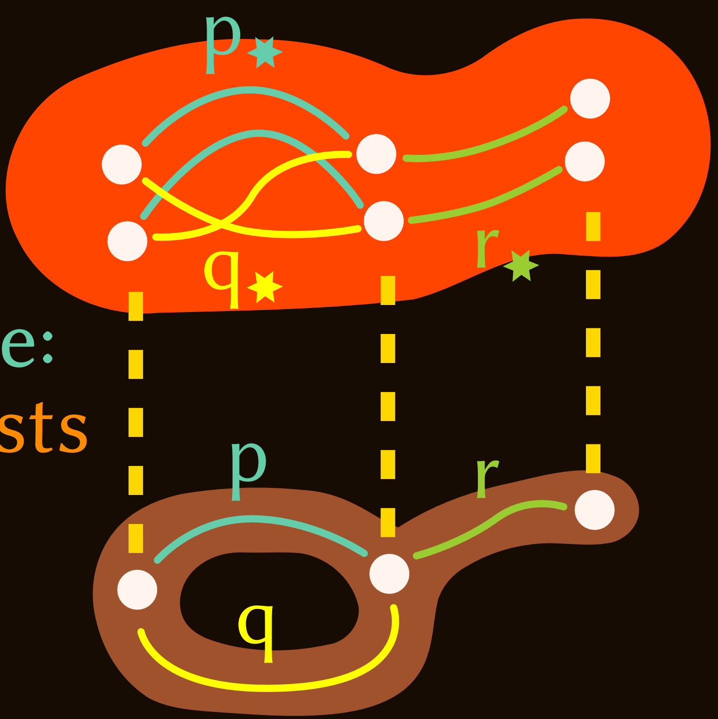
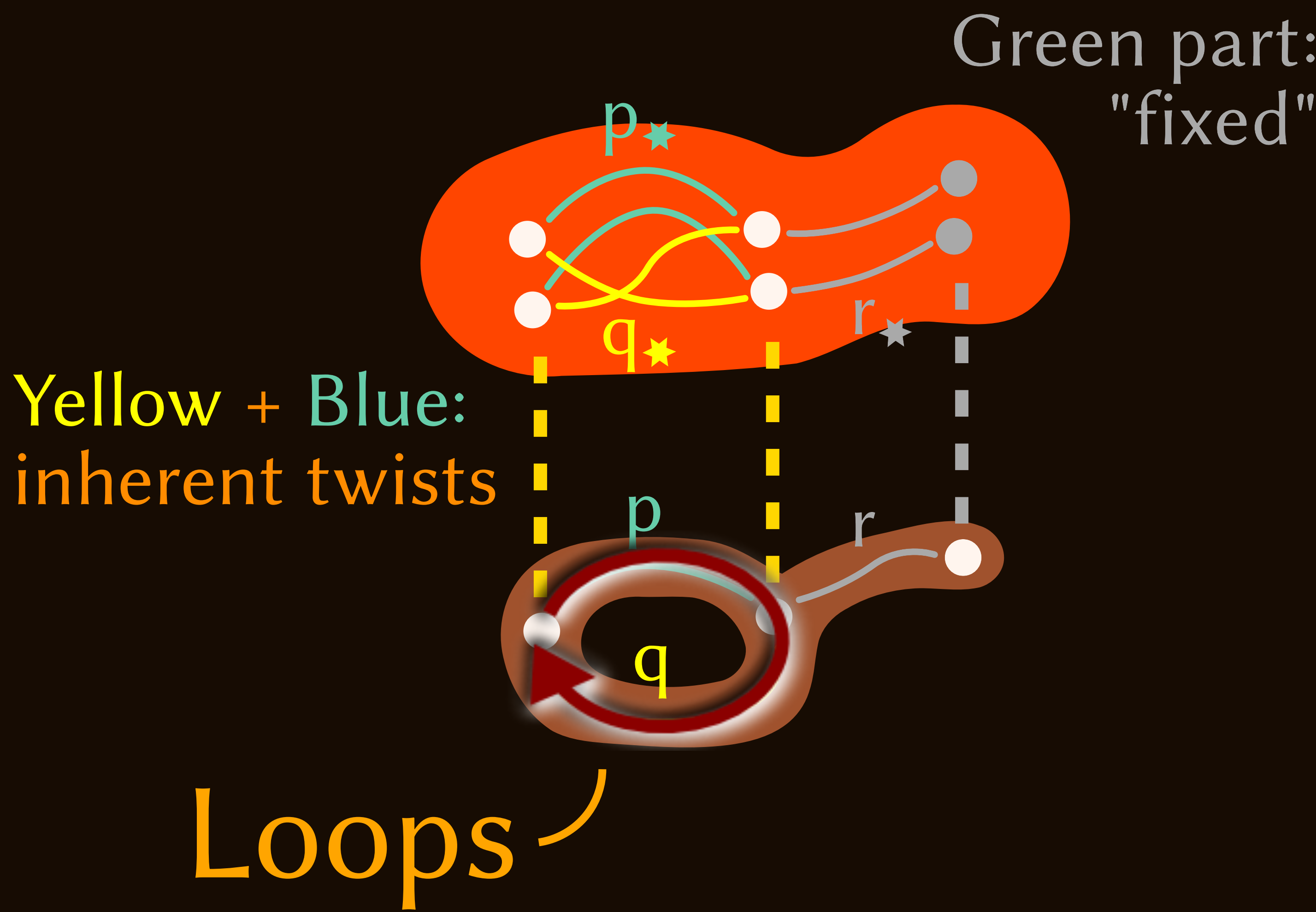
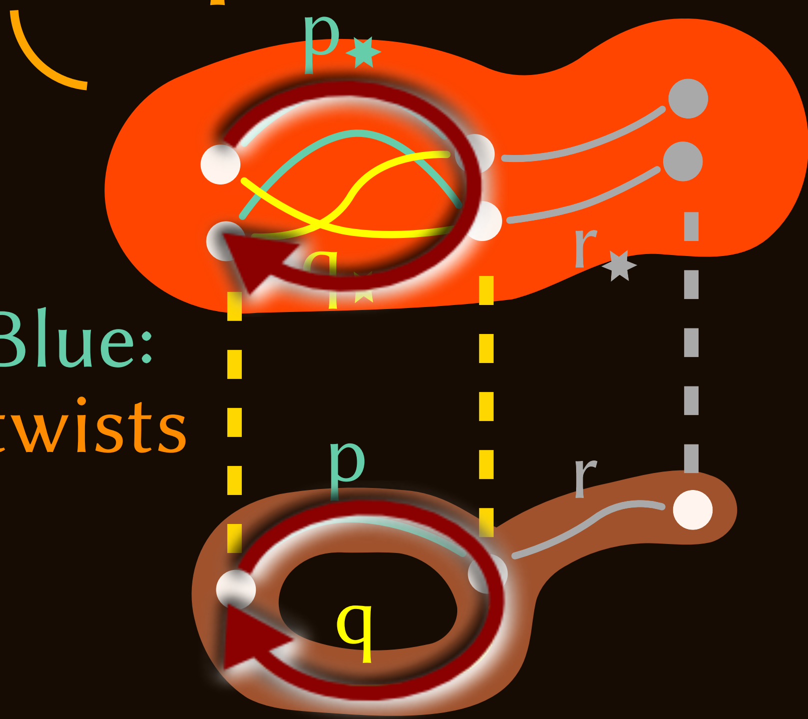




Green part:
"fixed"
p⋆
q⋆
r⋆
Yellow + Blue:
inherent twists
p
r
q
Loops

Automorphisms

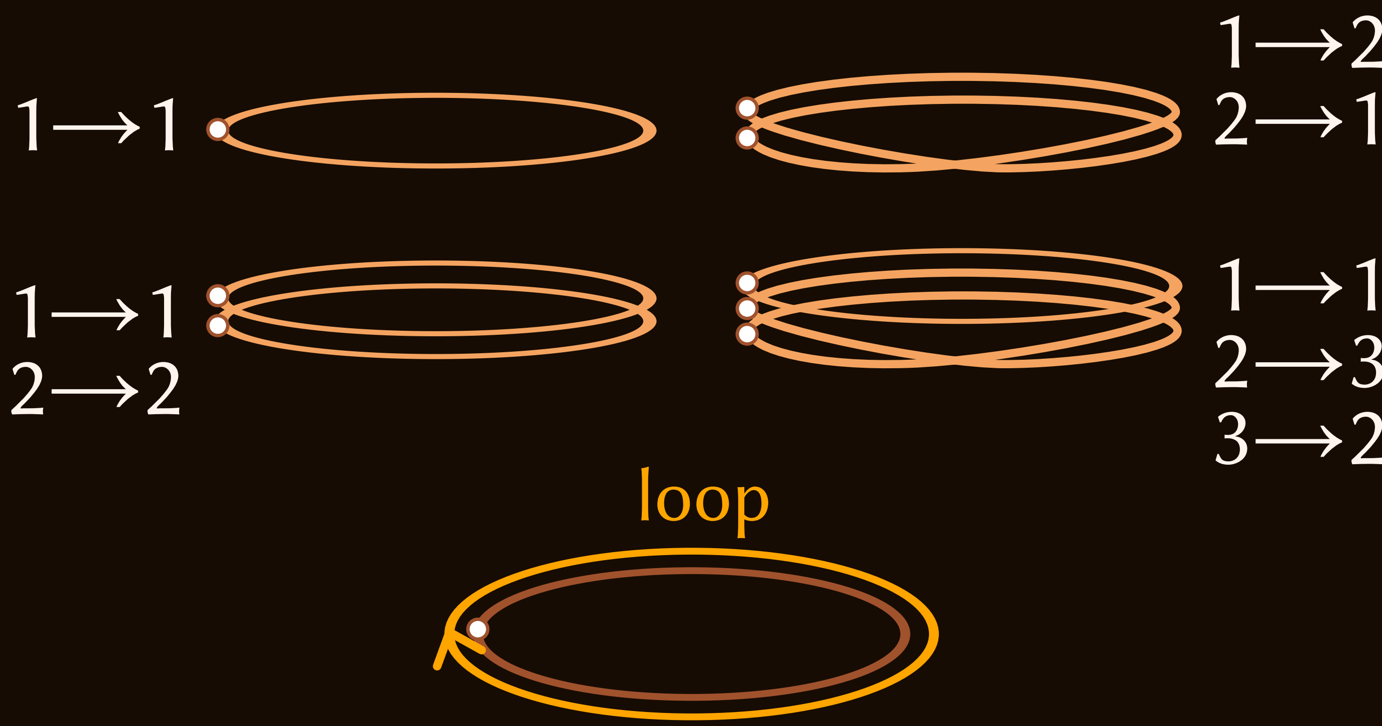
Green part: "fixed"



Yellow + Blue: inherent twists

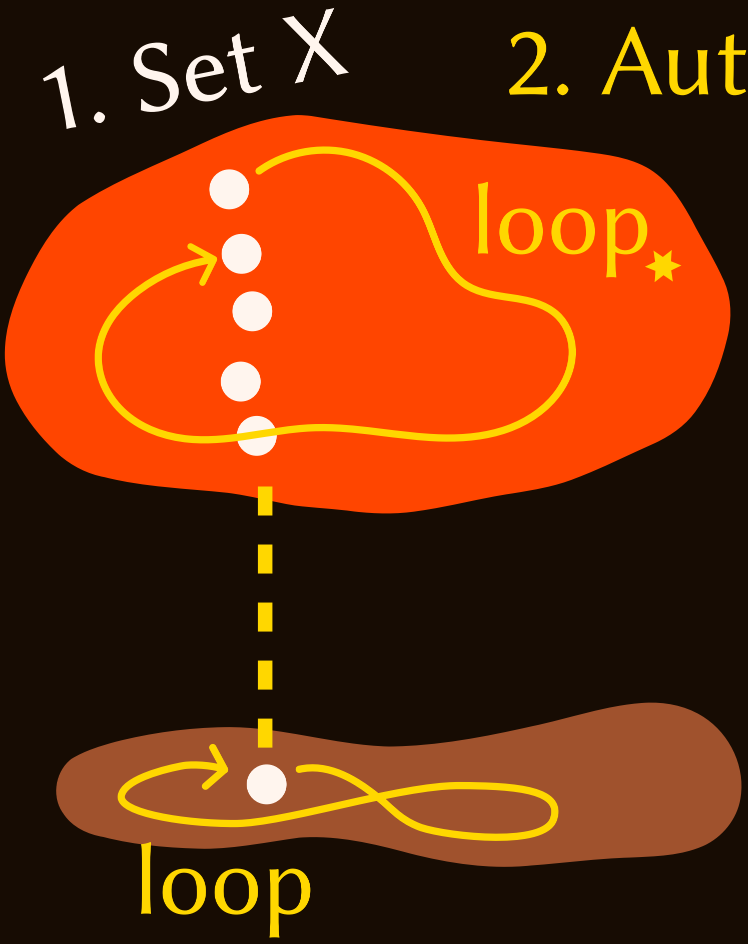
Loops

For circles...



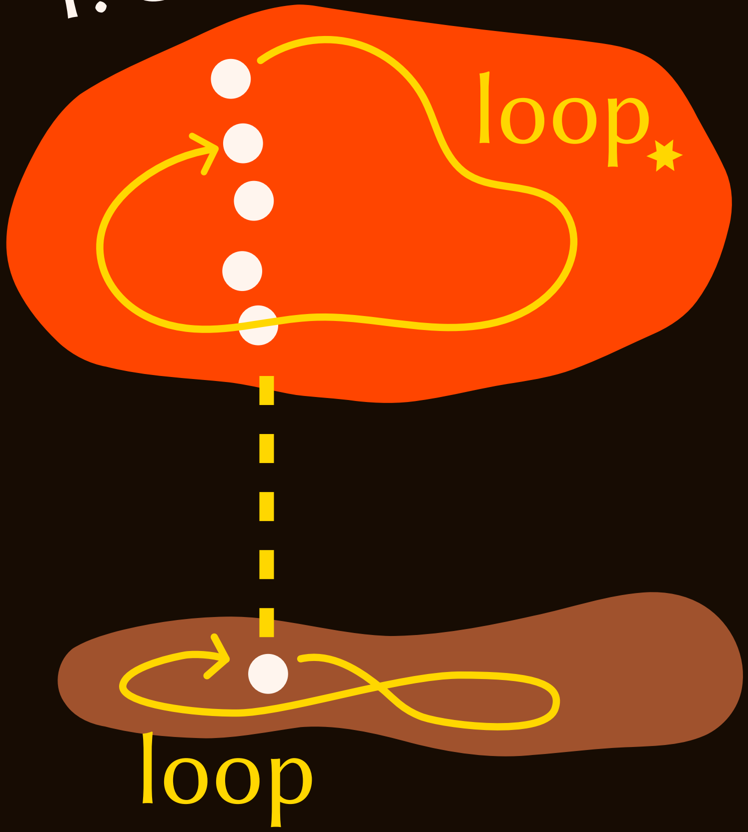
It is sufficient to check the generator loop

2. Automorphisms by different loops



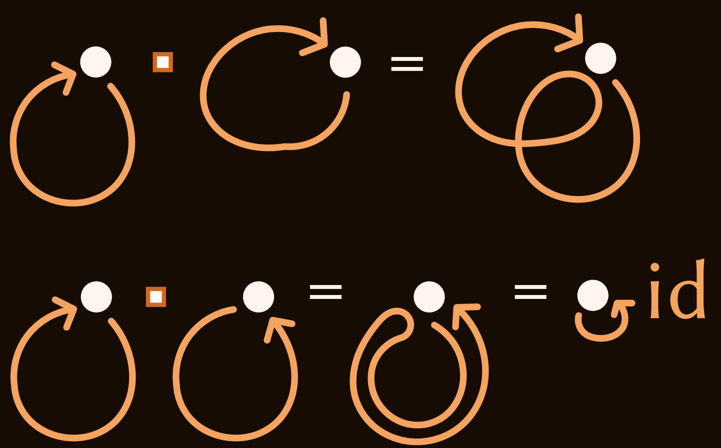
1. Set X

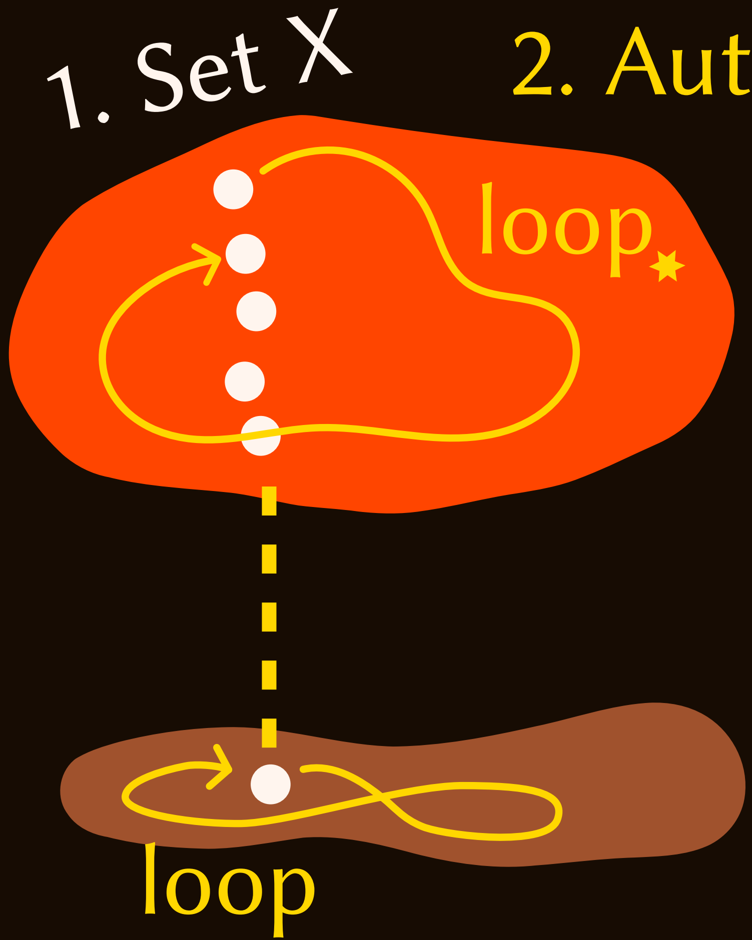
2. Automorphisms by different loops



Fundamental Group

Sets of loops based at a point





2. Automorphisms by different loops

elements in fundamental group

Fix G = fundamental group

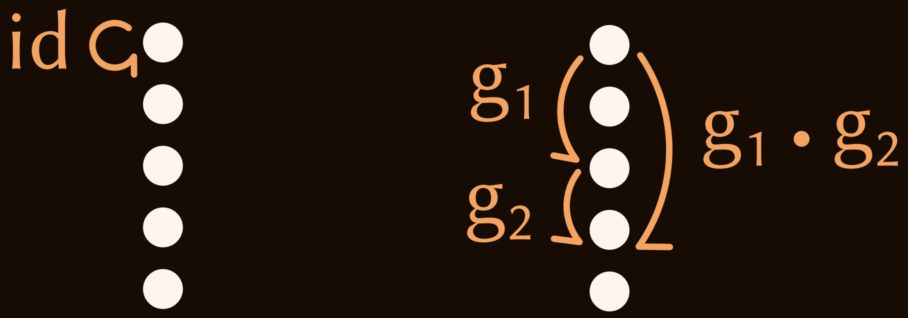
A G-set is a set X
with an action of G
)
map from G to automorphisms of X

Fix G = fundamental group

A G-set is a set X
with an action of G

map from G to automorphisms of X

with functoriality...



Fix G = fundamental group

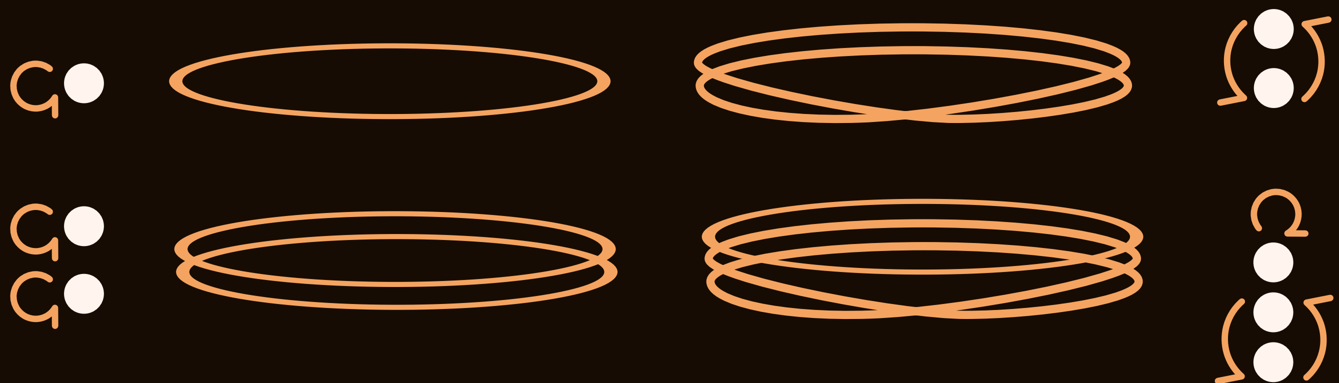
G-set

A set X equipped with an action, a map from G to automorphisms

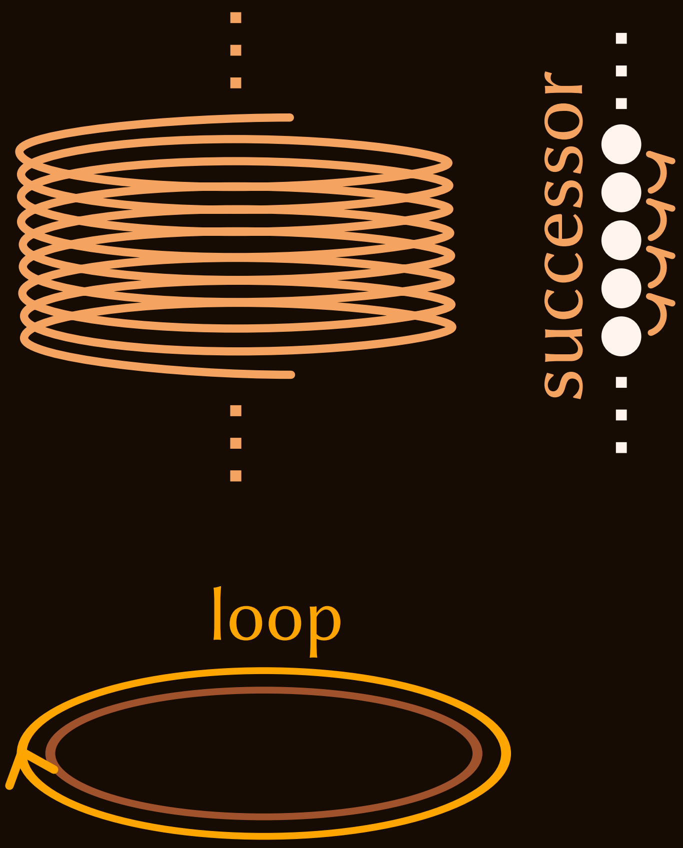
Classification Theorem

G-sets and covering spaces are *equivalent.*

For circles...

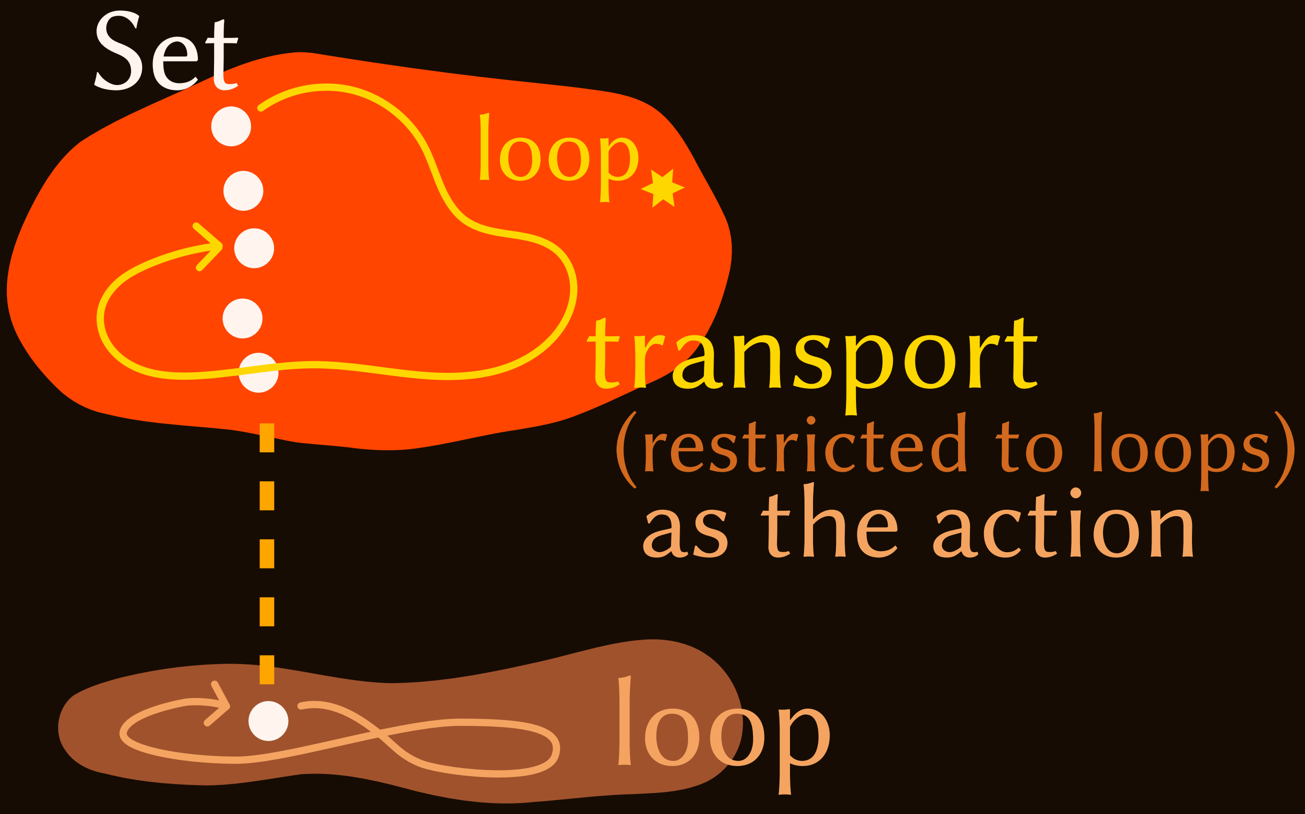


It is sufficient to check
the generator loop


successor
loop

$$\text{Cover} \xrightleftharpoons{\text{Proof}} \text{G-Set}$$

1. Cover ⟶ G-set



2. Cover ⟶ G-set ⟶ Cover

Given a G-set = a set X and an action

Construct a cover such that

1. Every fiber is isomorphic to X

2. Transport is the action
   (restricted to loops)

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Given a G-set = a set X and an action

Construct a cover such that

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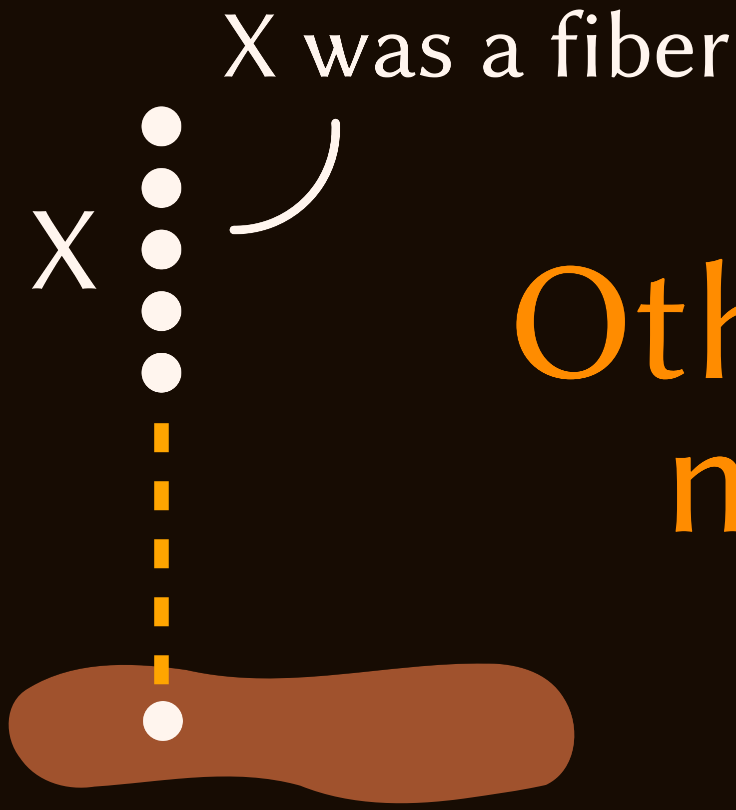
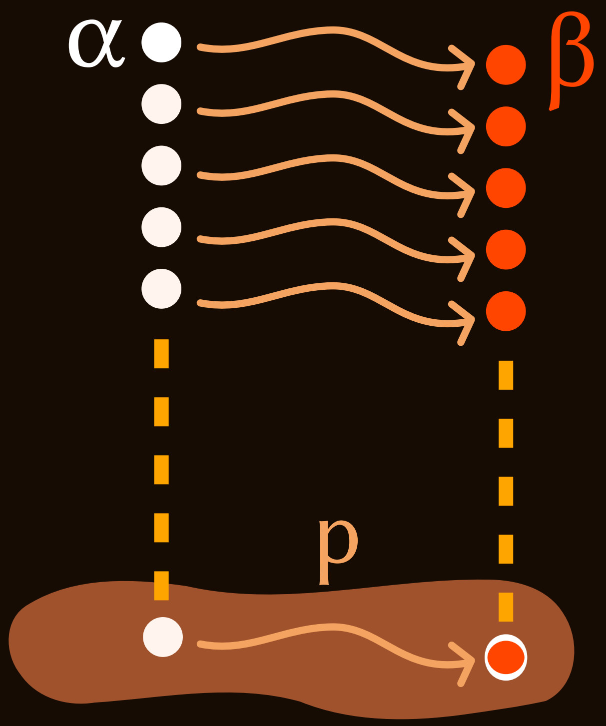
Magic: Higher inductive types

2. Cover ⟶ G-set ⟶ Cover



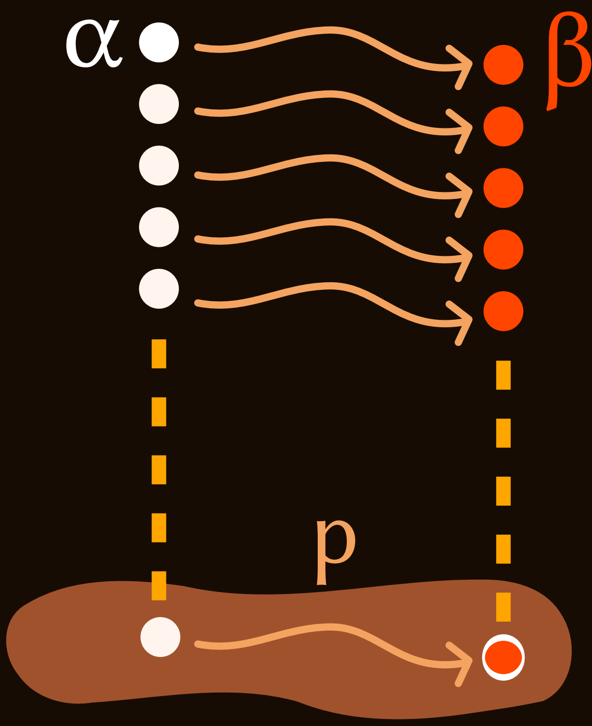
Other fibers missing

2. Cover ⟶ G-set ⟶ Cover



Base path p
would induce
an isomorphism
(by *"transport"*)

2. Cover ⟶ G-set ⟶ Cover

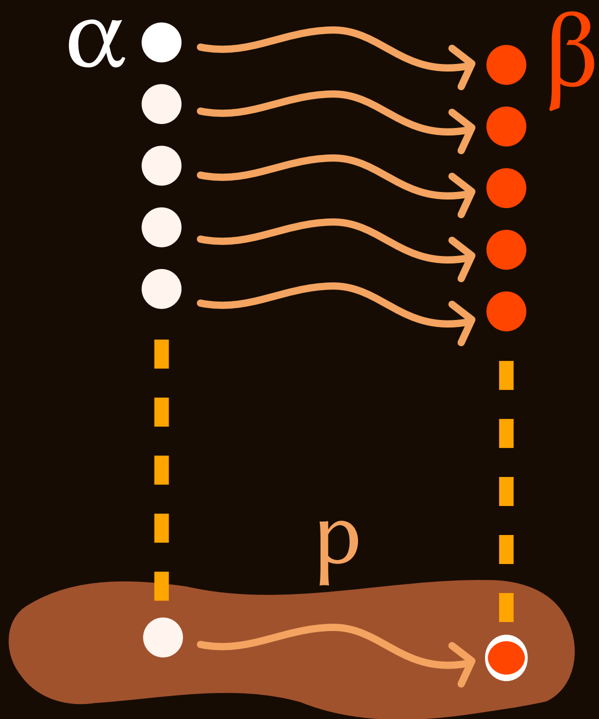


Base path p
would induce
an isomorphism
(by *"transport"*)

Fake it with
a formal one!

Point β is $p_{\star}\alpha$

2. Cover ⟶ G-set ⟶ Cover

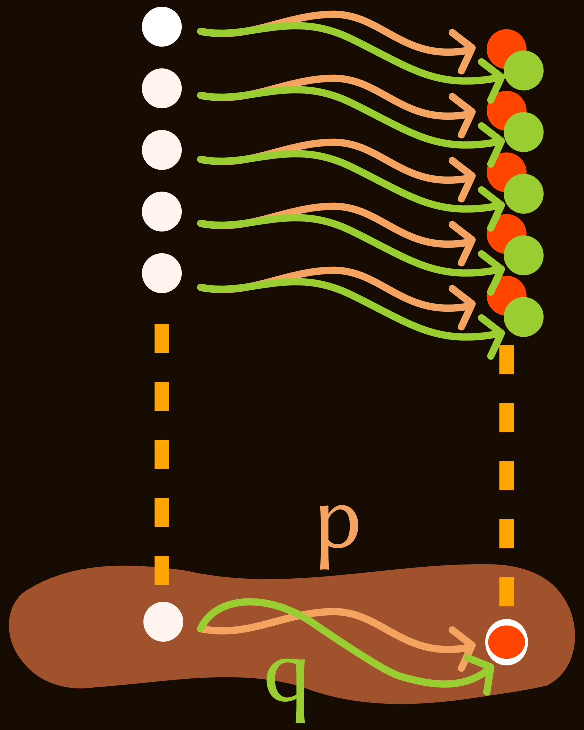



```
data R (a : A) : Set
  ✡ : ∀ p α → R a
```

formal transport

Point β is $p_{\text{✡}}\alpha$

2. Cover ⟶ G-set ⟶ Cover

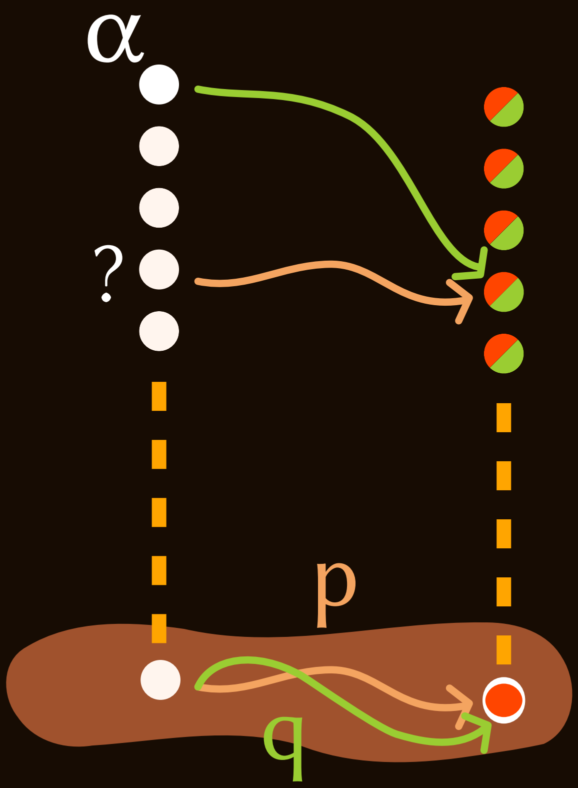


Different q's give different copies

Needs a way to *merge* copies from different base paths

2. Cover ⟶ G-set ⟶ Cover

If it will be some cover...



2. Cover ⟶ G-set ⟶ Cover

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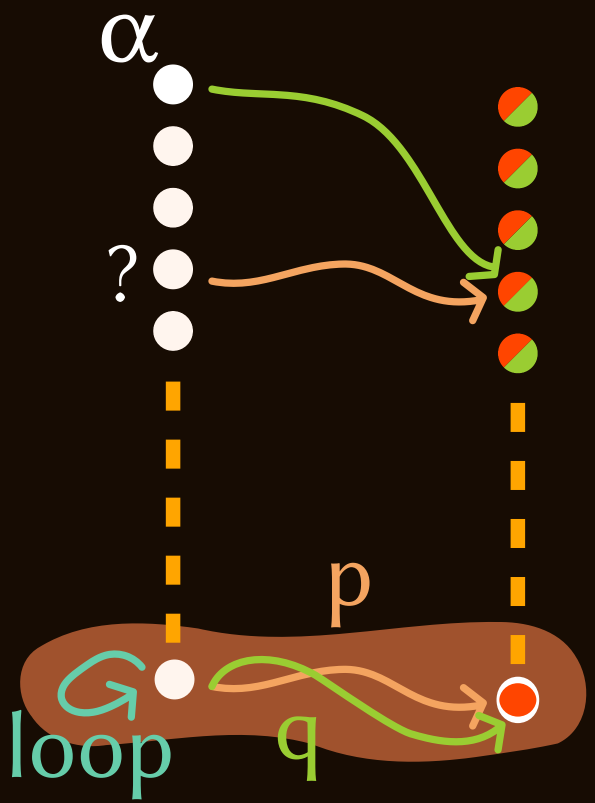


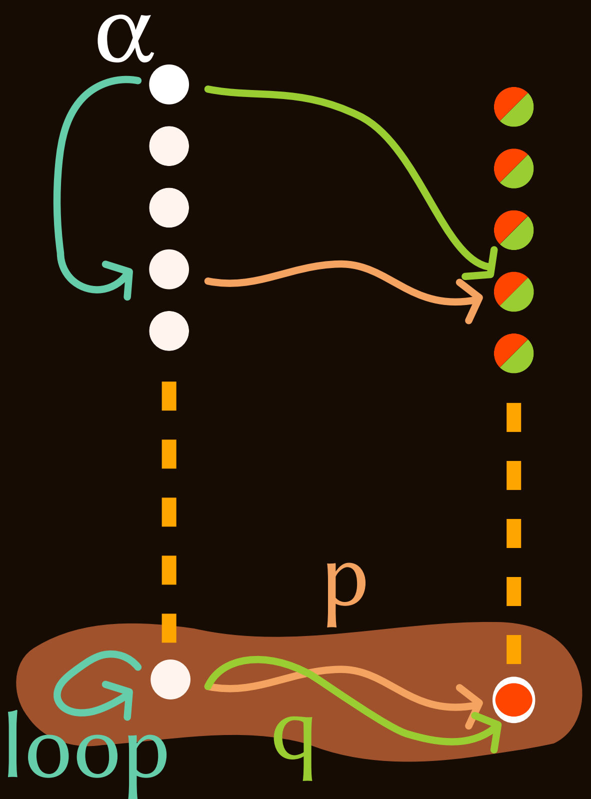
q must be $(q \cdot p^{-1}) \cdot p$

$\|$

loop

2. Cover ⟶ G-set ⟶ Cover

If it will be some cover...



q must be $(q \cdot p^{-1}) \cdot p$

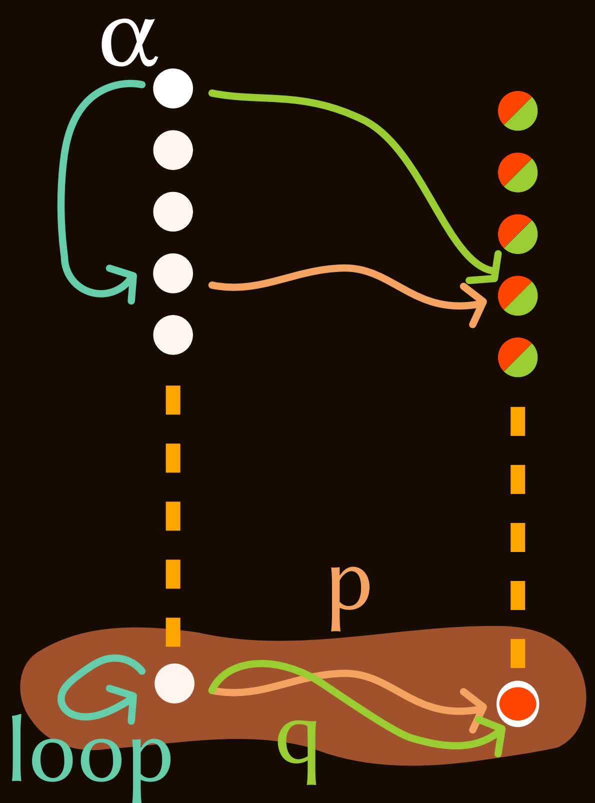
$$(q \cdot p^{-1}) = \text{loop}$$

$$\begin{aligned} & q \star \alpha \\ = & (\text{loop} \cdot p) \star \alpha \\ = & p \star (\text{loop} \star \alpha) \end{aligned}$$

Key: functoriality

2. Cover ⟶ G-set ⟶ Cover

Going back to the construction...



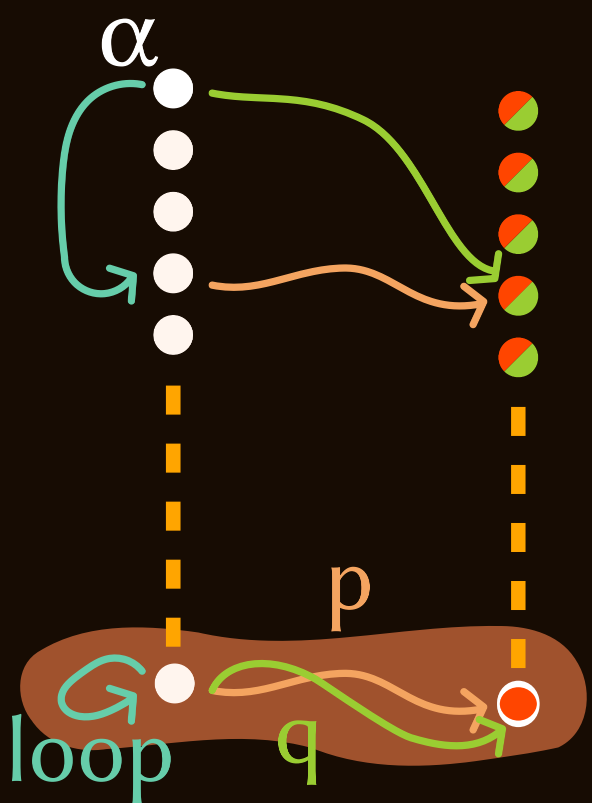
We mimic functoriality

$$
\begin{aligned}
& q \mathbin{\davidsstar} \alpha \\
= {} & (\mathrm{loop} \cdot p) \mathbin{\davidsstar} \alpha \\
= {} & p \mathbin{\davidsstar} (\mathrm{loop} \mathbin{\davidsstar} \alpha)
\end{aligned}
$$

action is transport for loops

$$
\begin{aligned}
& q \star \alpha \\
= {} & (\mathrm{loop} \cdot p) \star \alpha \\
= {} & p \star (\mathrm{loop} \star \alpha)
\end{aligned}
$$

2. Cover ⟶ G-set ⟶ Cover



We mimic functoriality

q ✡ α
= (loop ▪ p) ✡ α
= p ✡ (loop ✡ α)

```
data R (a : A) : Set
  ✡ : ∀ p α → R a
  ● : ∀ l p α
        → (l ▪ p) ✡ α
        = p ✡ (l ✡ α)
```

2. Cover ⟶ G-set ⟶ Cover

```
data R (a : A) : Set
  ✡ : ∀ p α → R a
  ● : ∀ l p α → (l ▪ p) ✡ α = p ✡ (l ✡ α)
```

Theorem

R is equivalent to the original cover

Acknowledgements:
Thanks to Guillaume Brunerie,
Daniel Grayson and Chris Kapulkin
for helping me state and prove this.

[recap]

Classification Theorem

If G is the fundamental group G-sets and covering spaces are equivalent.

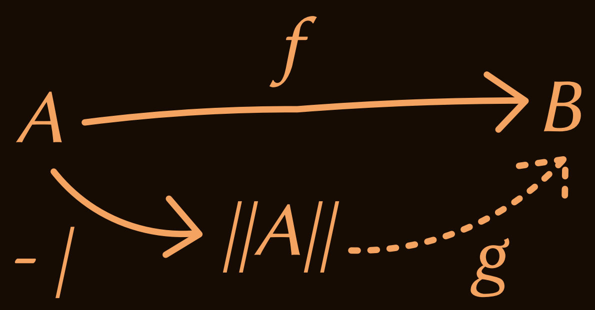
Technical Notes

WARNING: NASTY MATH AHEAD

All *truncations* were omitted.

You want this lemma:

Given a constant (pointwise-equal) function $f : A \longrightarrow B$ where B is a set find a $g : \|A\| \longrightarrow B$ such that $f = g \cdot |\text{-}|$

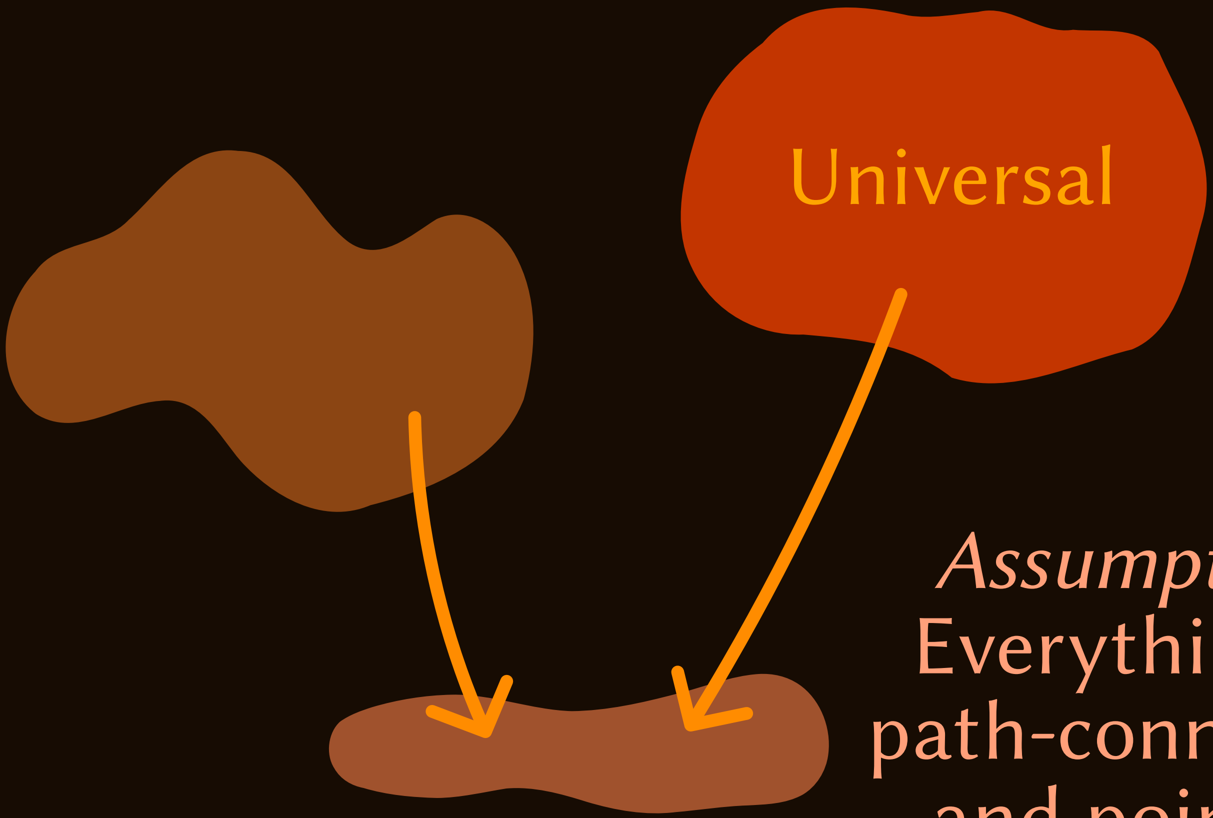


Part 2

Universality

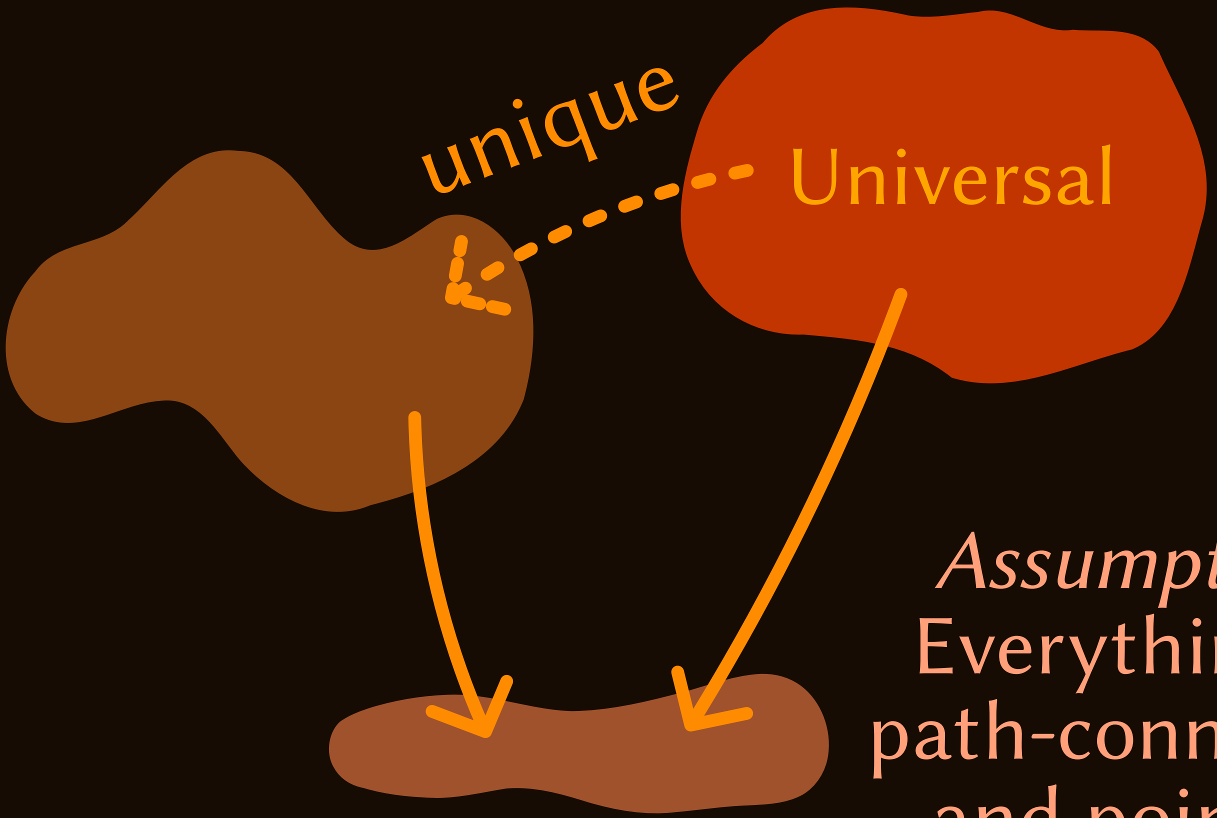
covers that cover every cover

Universality



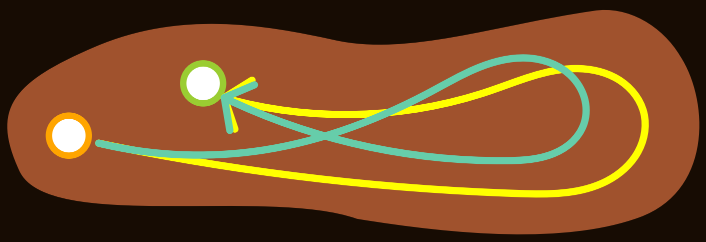
Assumption: Everything is path-connected and pointed

Universality



Assumption: Everything is path-connected and pointed

A simple universal cover



set of paths
with one end fixed

● pointed

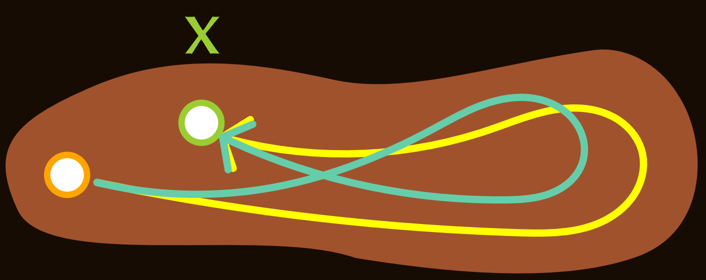
Assumption:
Everything is pointed
and path-connected.

Technical Notes

WARNING: NASTY MATH AHEAD

The simple universal cover is

$$\lambda\, x\, .\, \| \circ = x \|_0$$



set of paths
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Technical Notes

WARNING: NASTY MATH AHEAD

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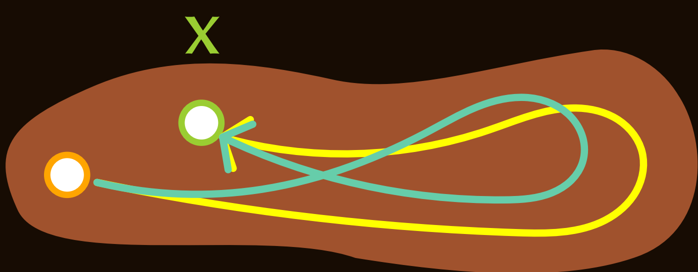
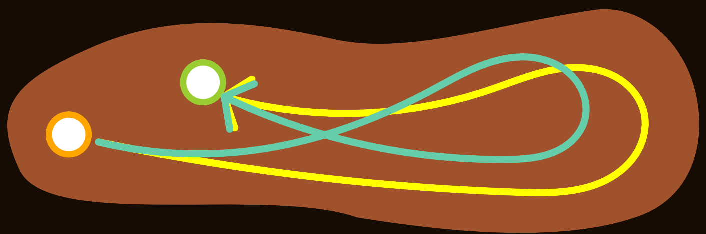


set of paths
with one end fixed

Path induction!

Theorem

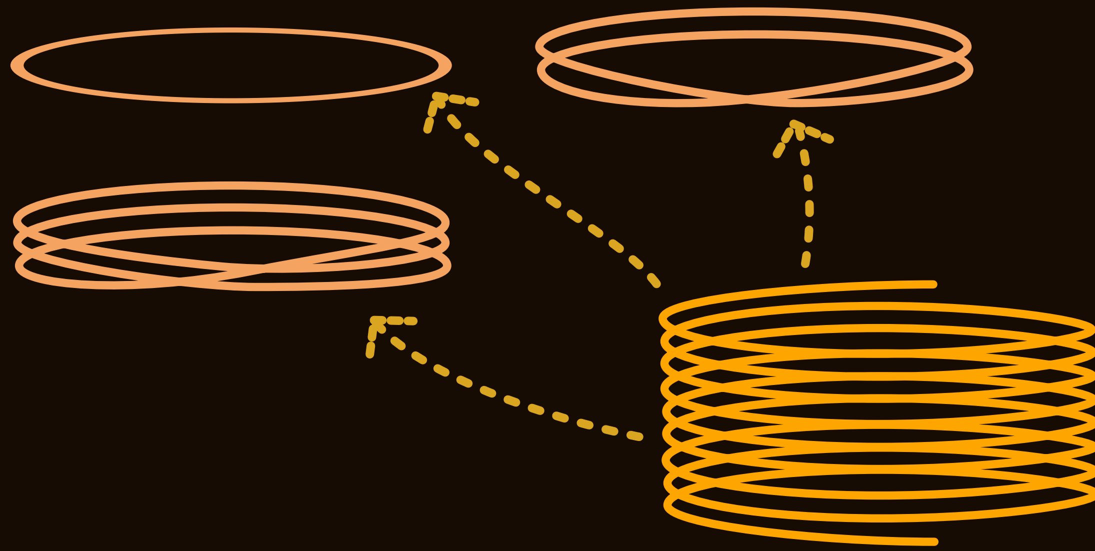


- It is inital.
- It is equivalent to any simply connected cover.

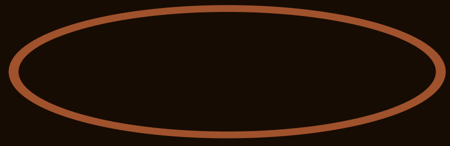
Simply Connected

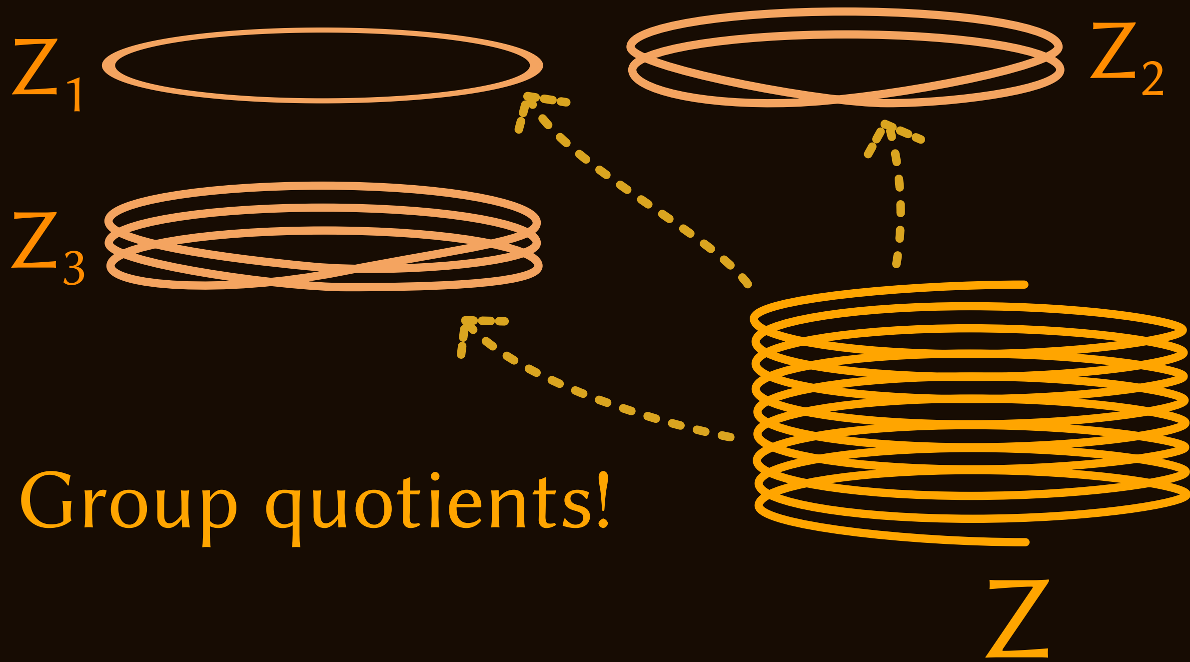
one and only one

circle



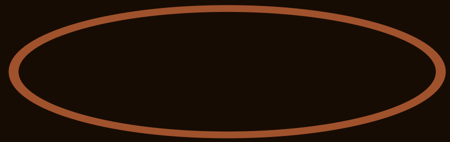
circle



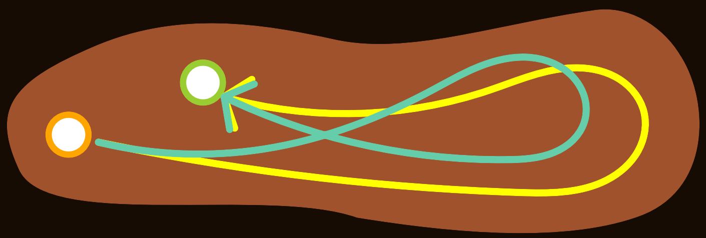


Group quotients!

circle



Theorem

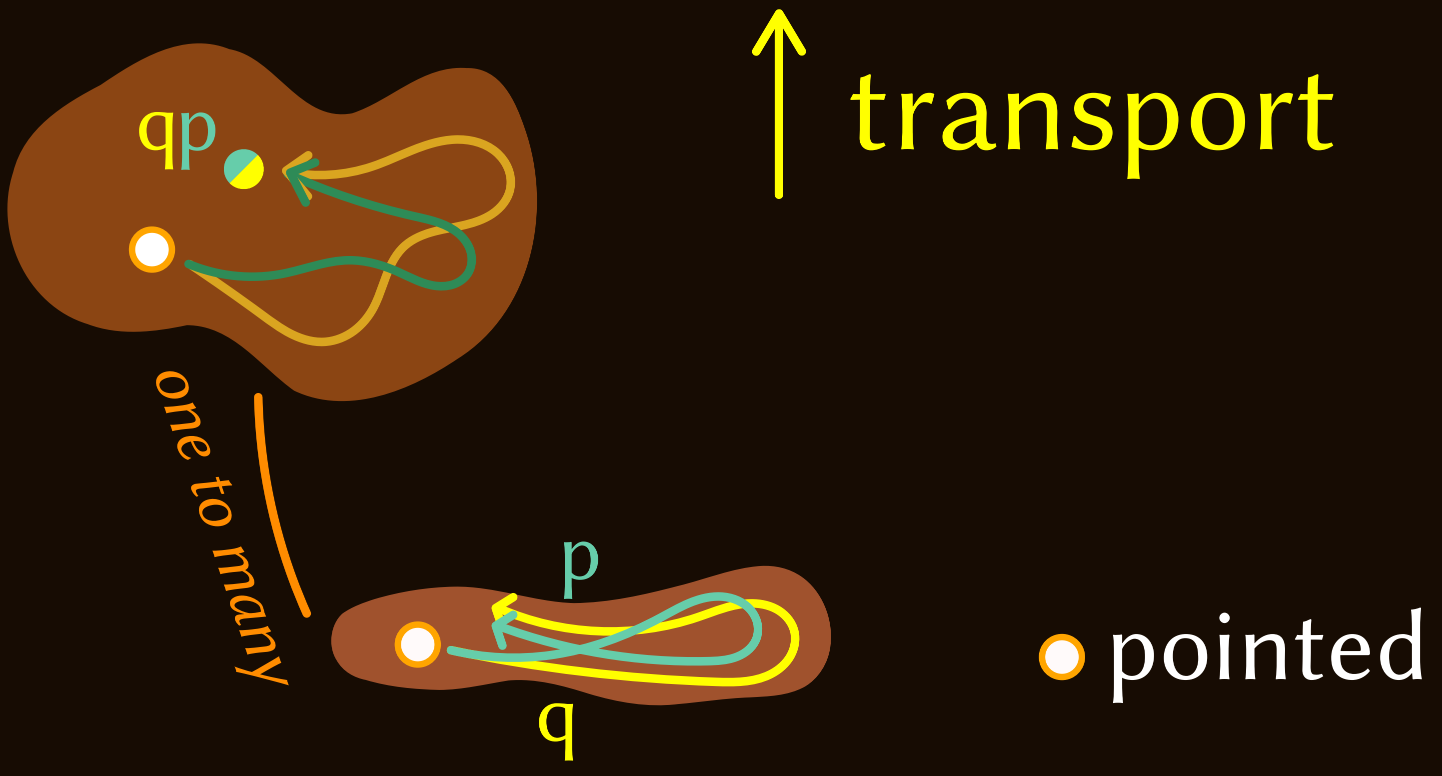


- It is inital.
- It is equivalent to any simply connected cover.

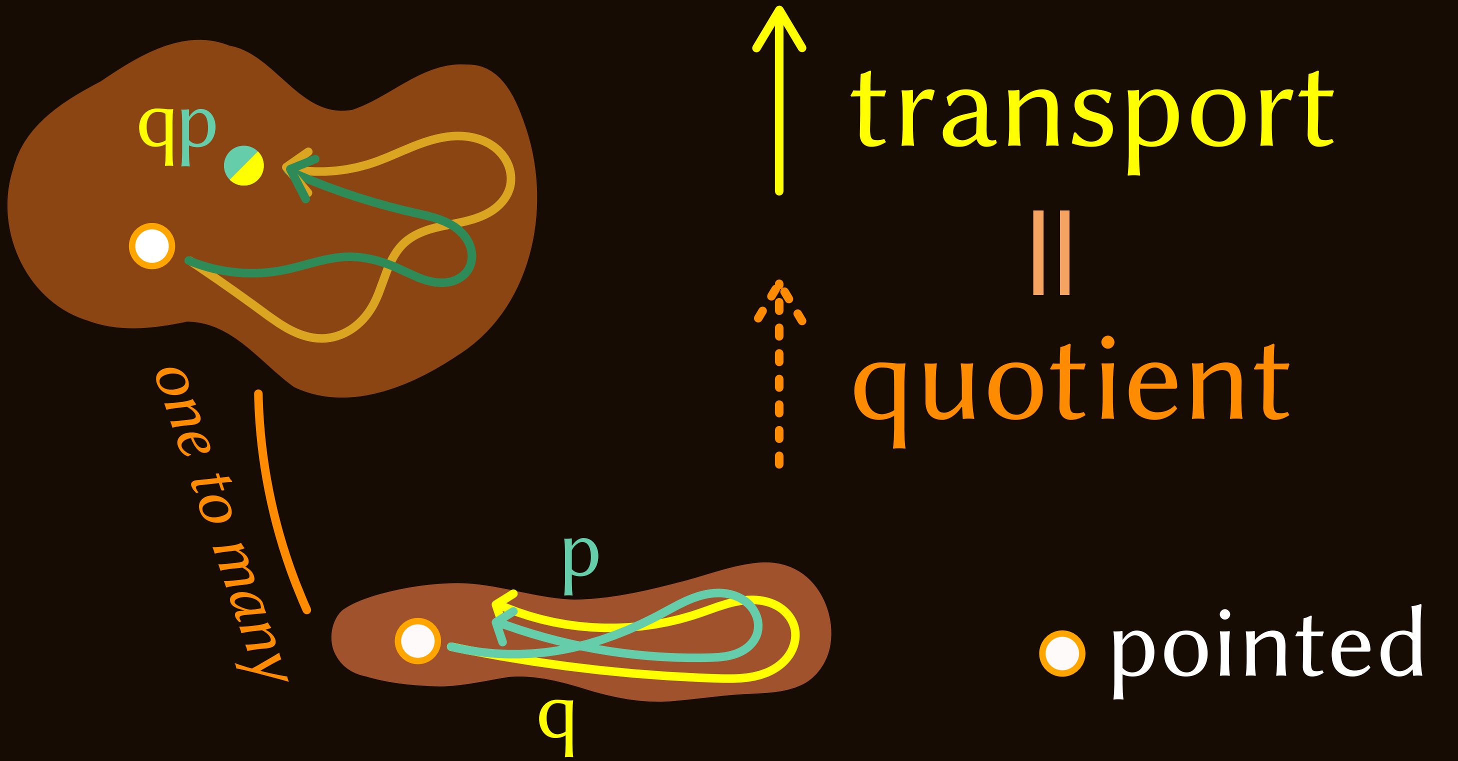
Simply Connected

one and only one

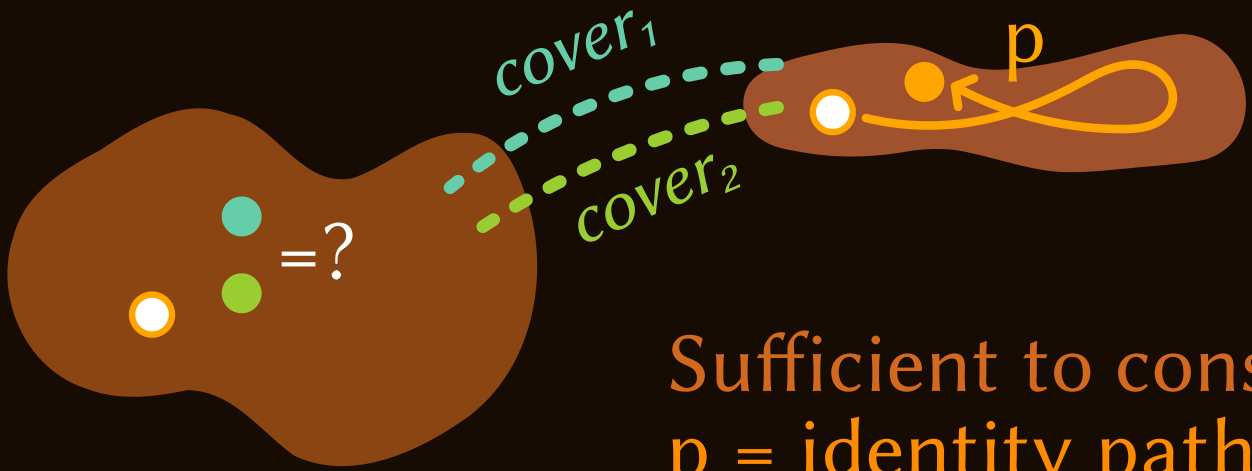
Weak Initiality



Weak Initiality

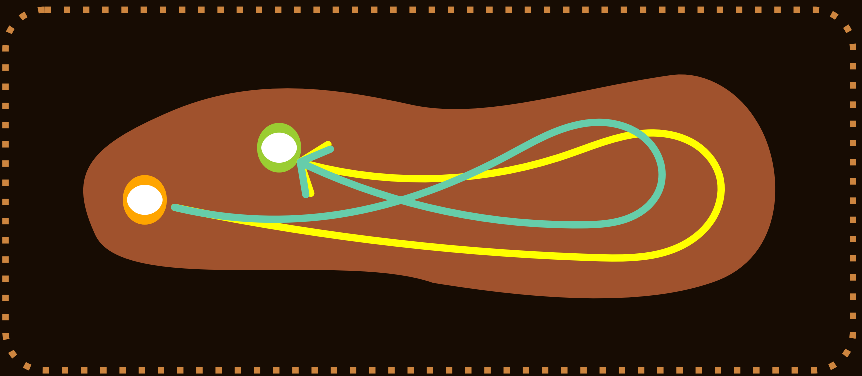


Strong Initiality



Sufficient to consider
p = identity path
(○ and ● collide)

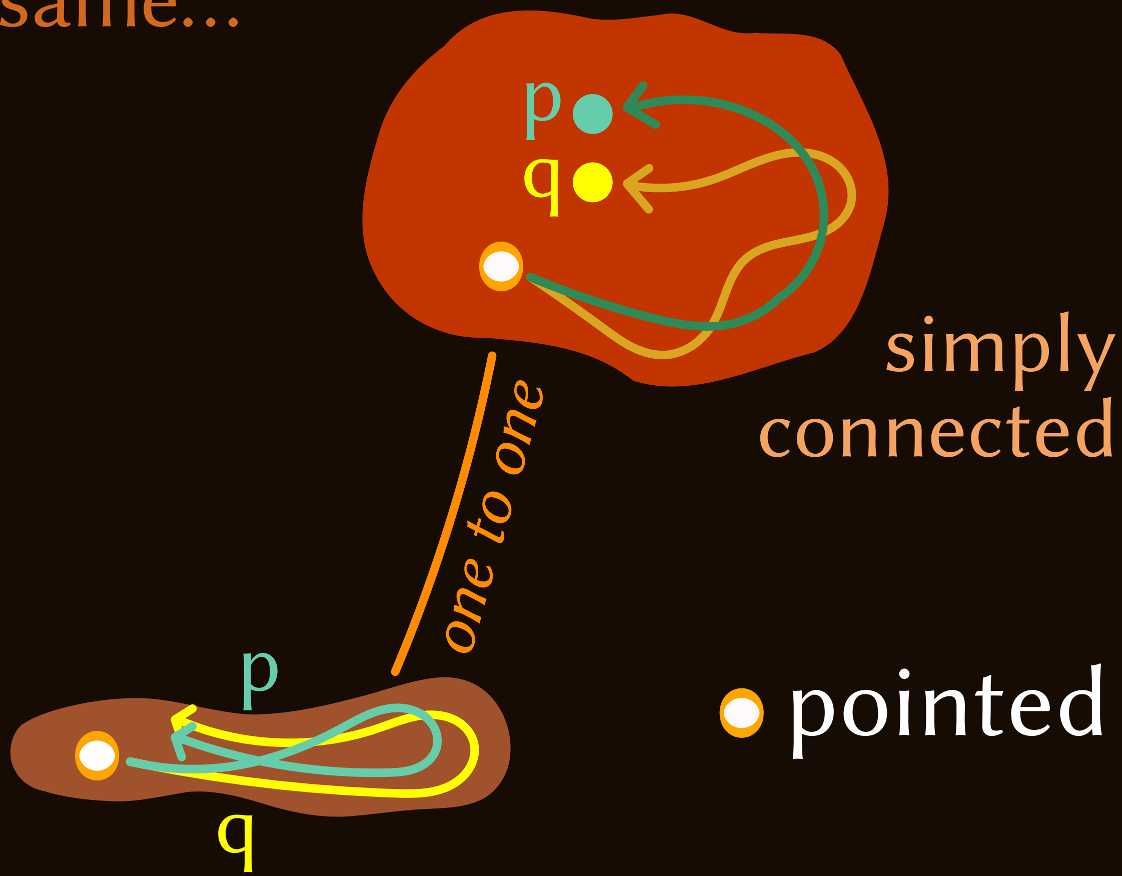
○ pointed



Theorem

- It is inital.
- It is equivalent to any simply connected cover.

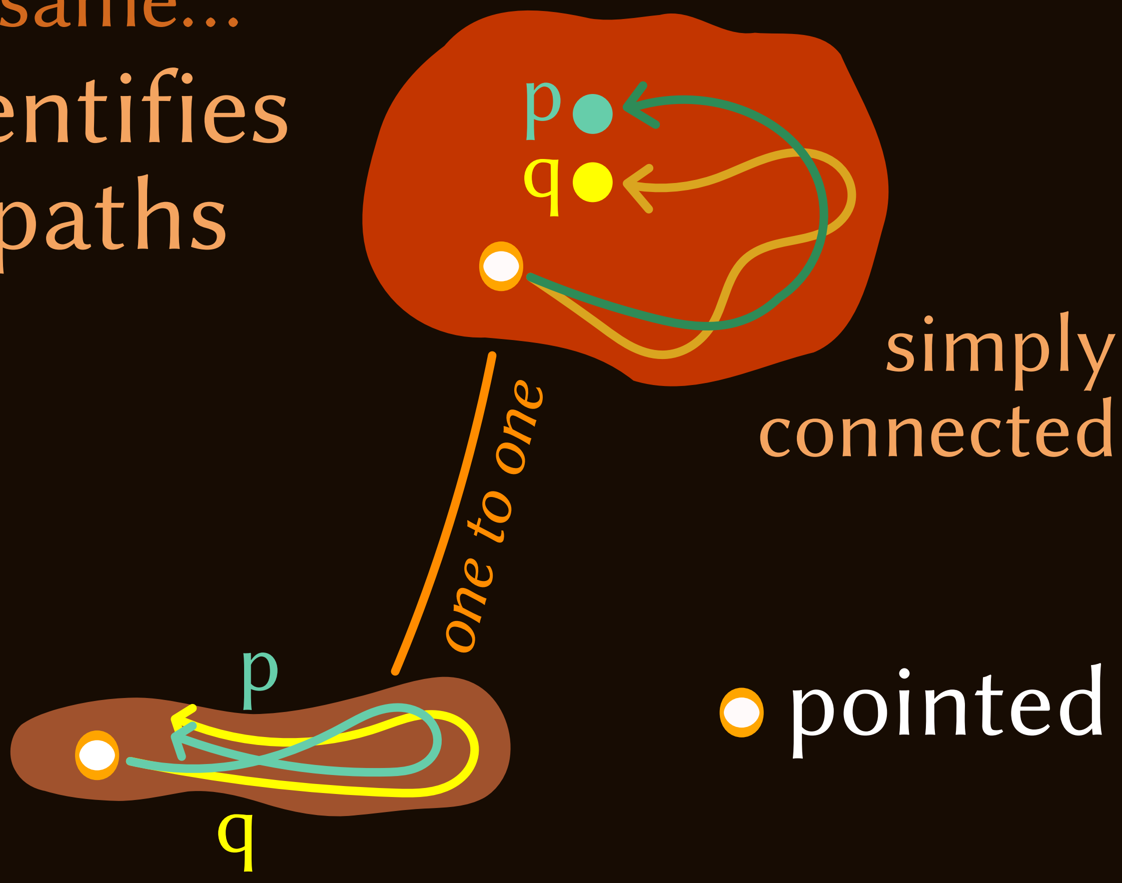
↑ If lifted p and q are the same...



↑ If lifted p and q are the same…

= *s.c.* identifies

↑ lifted paths



↑ If lifted p and q are the same...

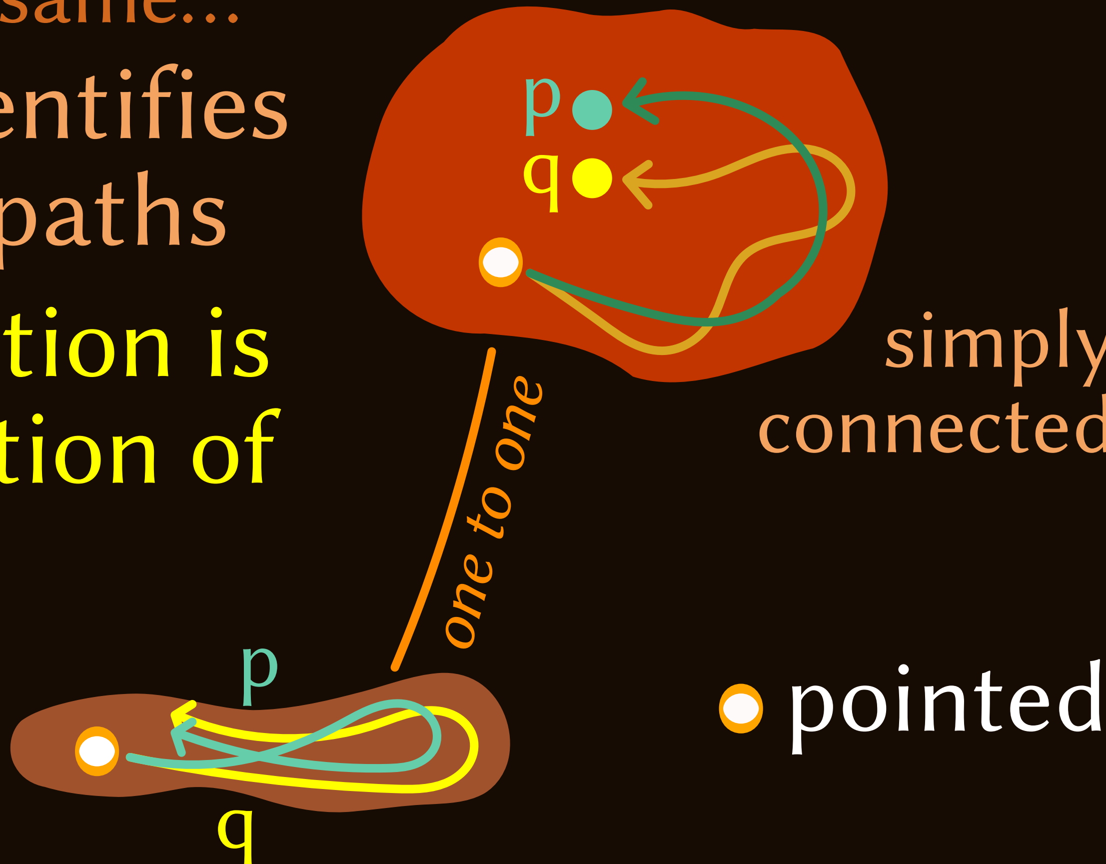
= *s.c.* identifies

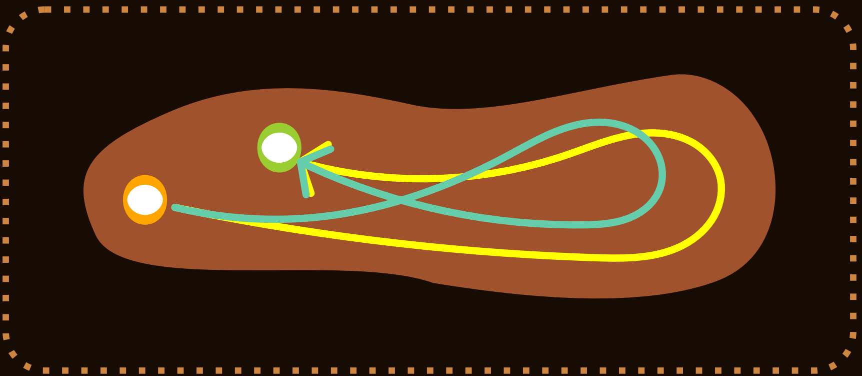
↑ lifted paths

= projection is

⇅ retraction of

= lifting





Theorem

- It is inital.
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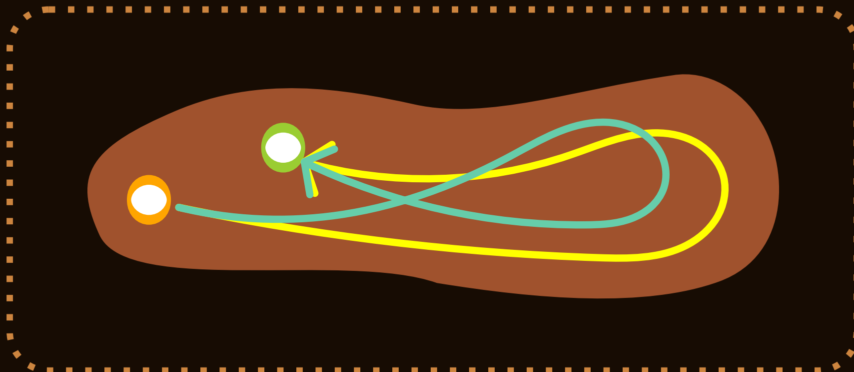
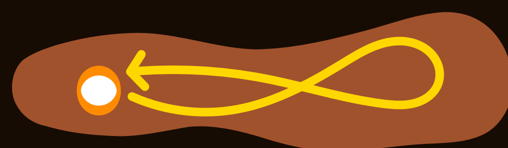


fiber over



||

fundamental group

Agda code

`github.com/HoTT/HoTT-Agda/blob/2.0`

Thanks

Agda code

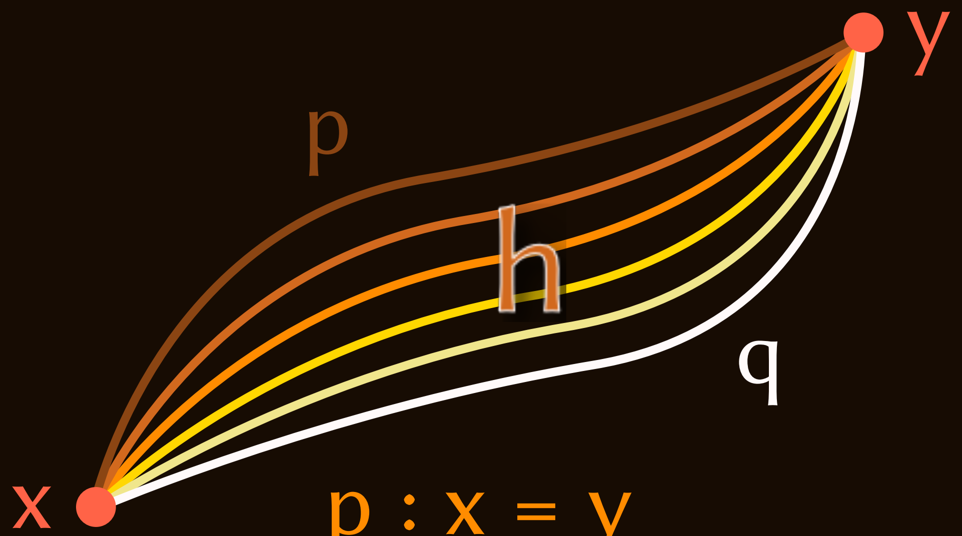
`github.com/HoTT/HoTT-Agda/blob/2.0`

Definition of Path Homotopy

continuous
deformation

Path of paths

Definition of Path Homotopy



$p : x = y$
$q : x = y$
$h : p = q$

Identity of identity types