Seifert-van Kampen Theorem in Homotopy Type Theory

[Toronto version]

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* **Favonia @ CMU**

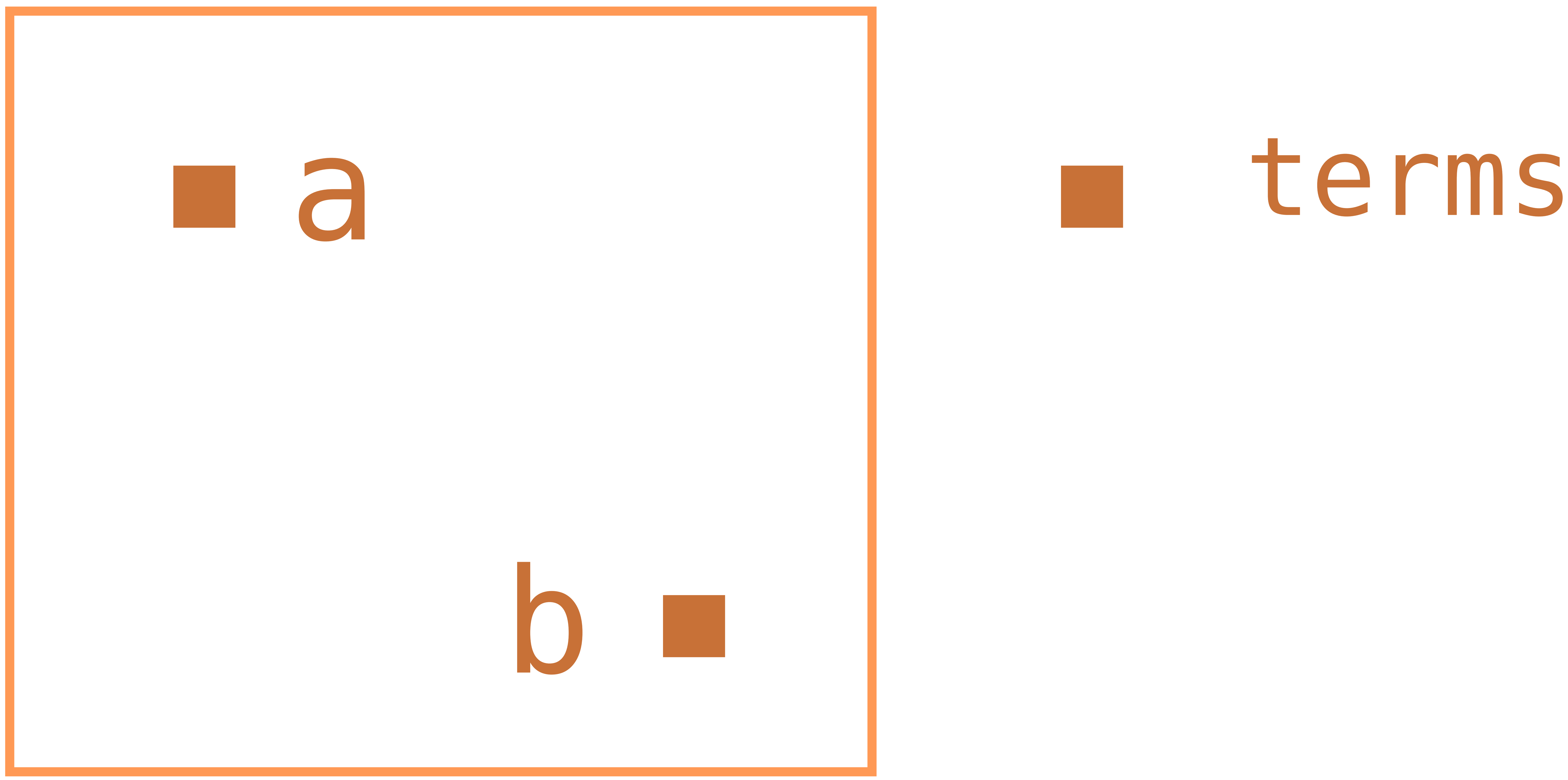
Homotopy Type Theory

* Type theory <-> topology

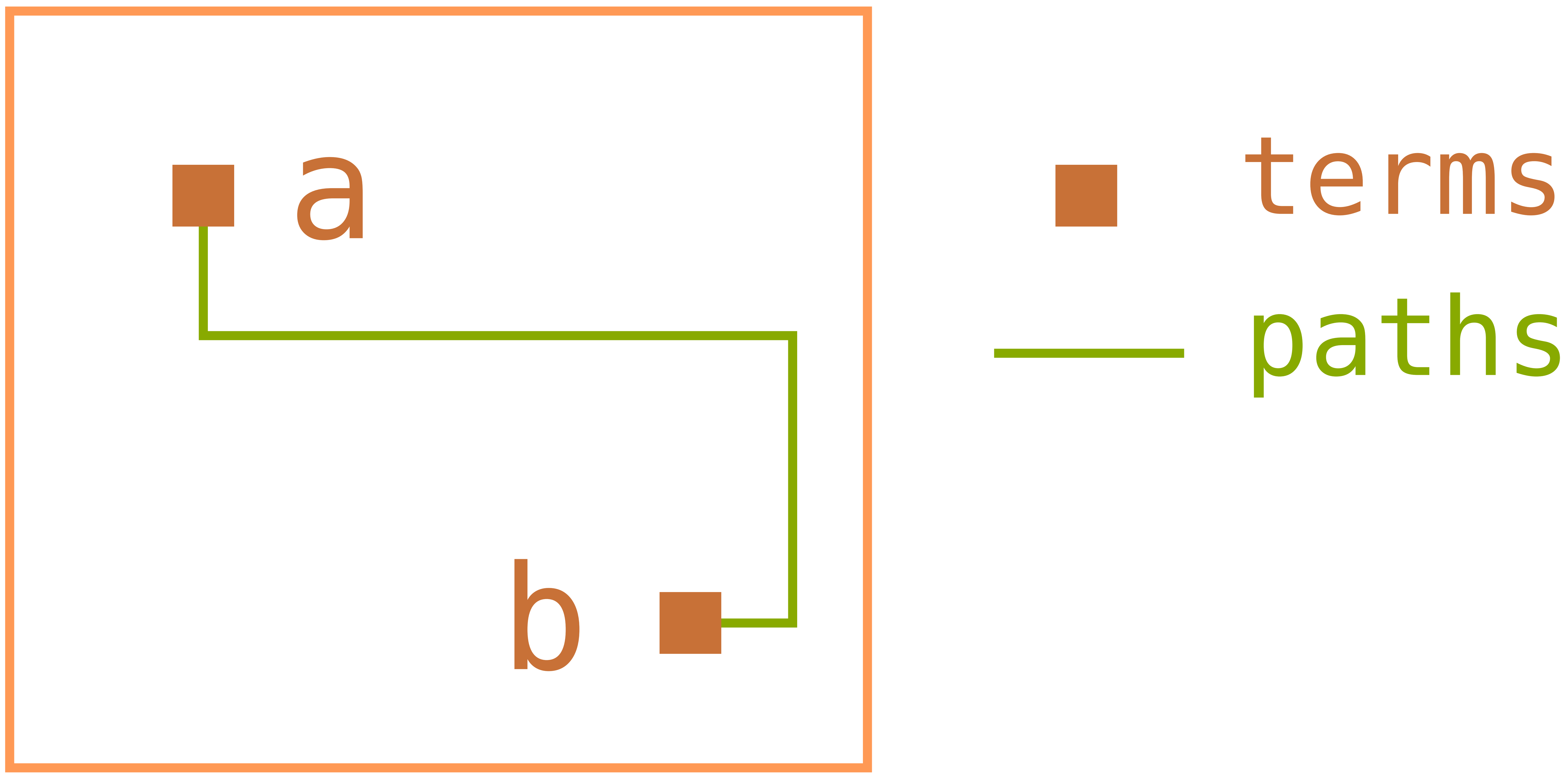
  - types ~= spaces
  - terms ~= points
  - functions ~= continuous maps
  - identifications ~= paths

* Non-trivial identifications

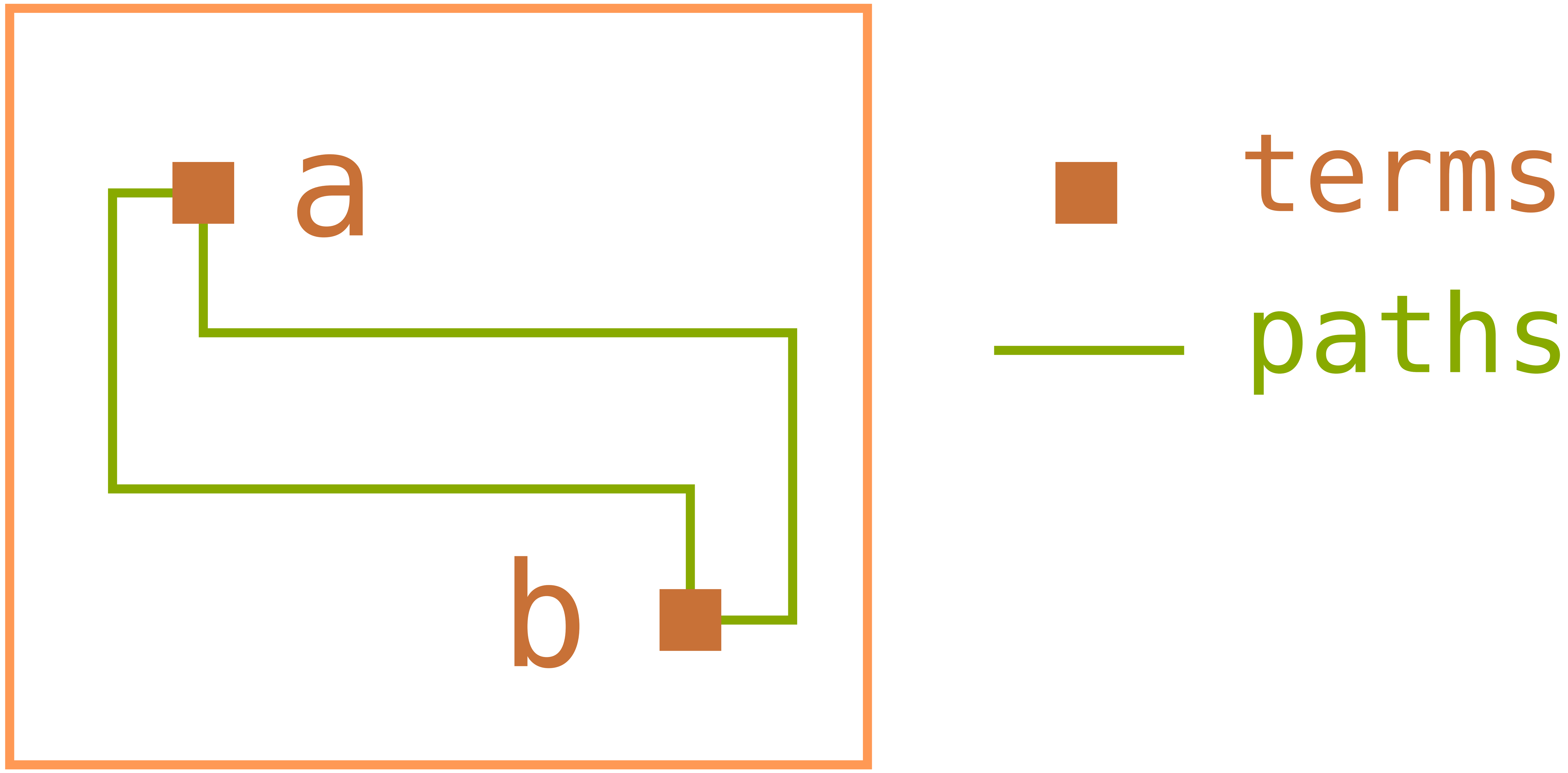
Iterated Paths



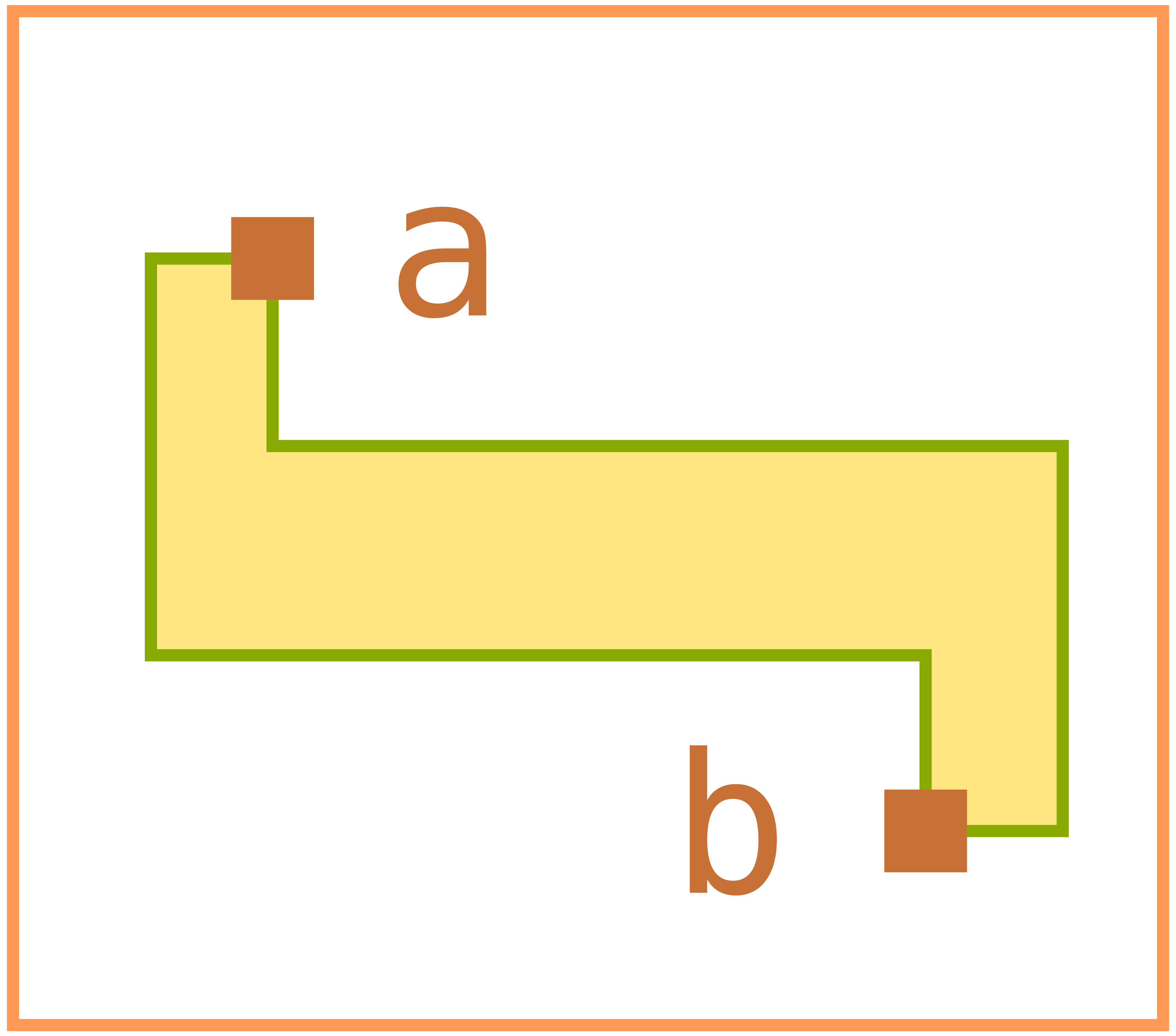
Iterated Paths



Iterated Paths



Iterated Paths

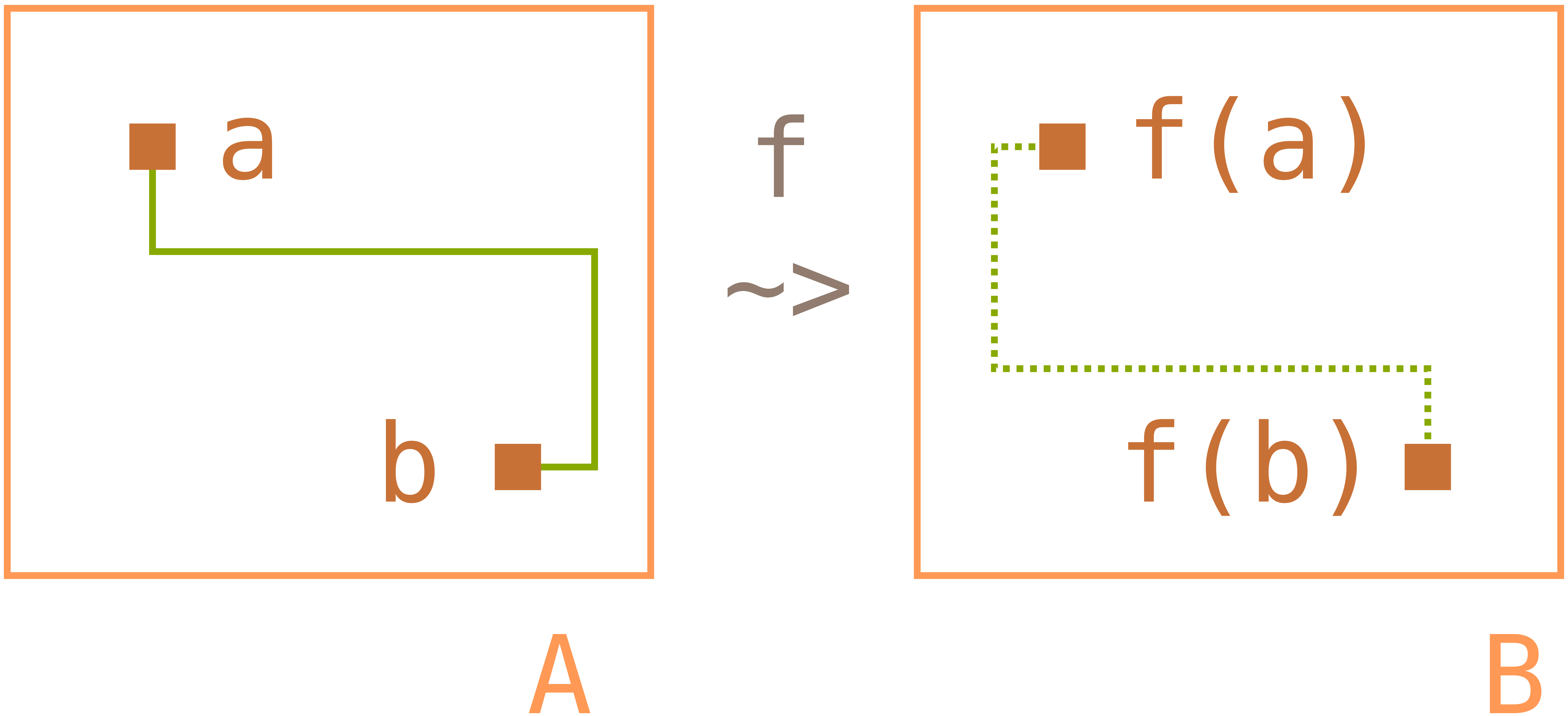


terms

paths

paths of paths

Functorial



Subject of Study

fundamental groups of pushouts

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"structure of loops"

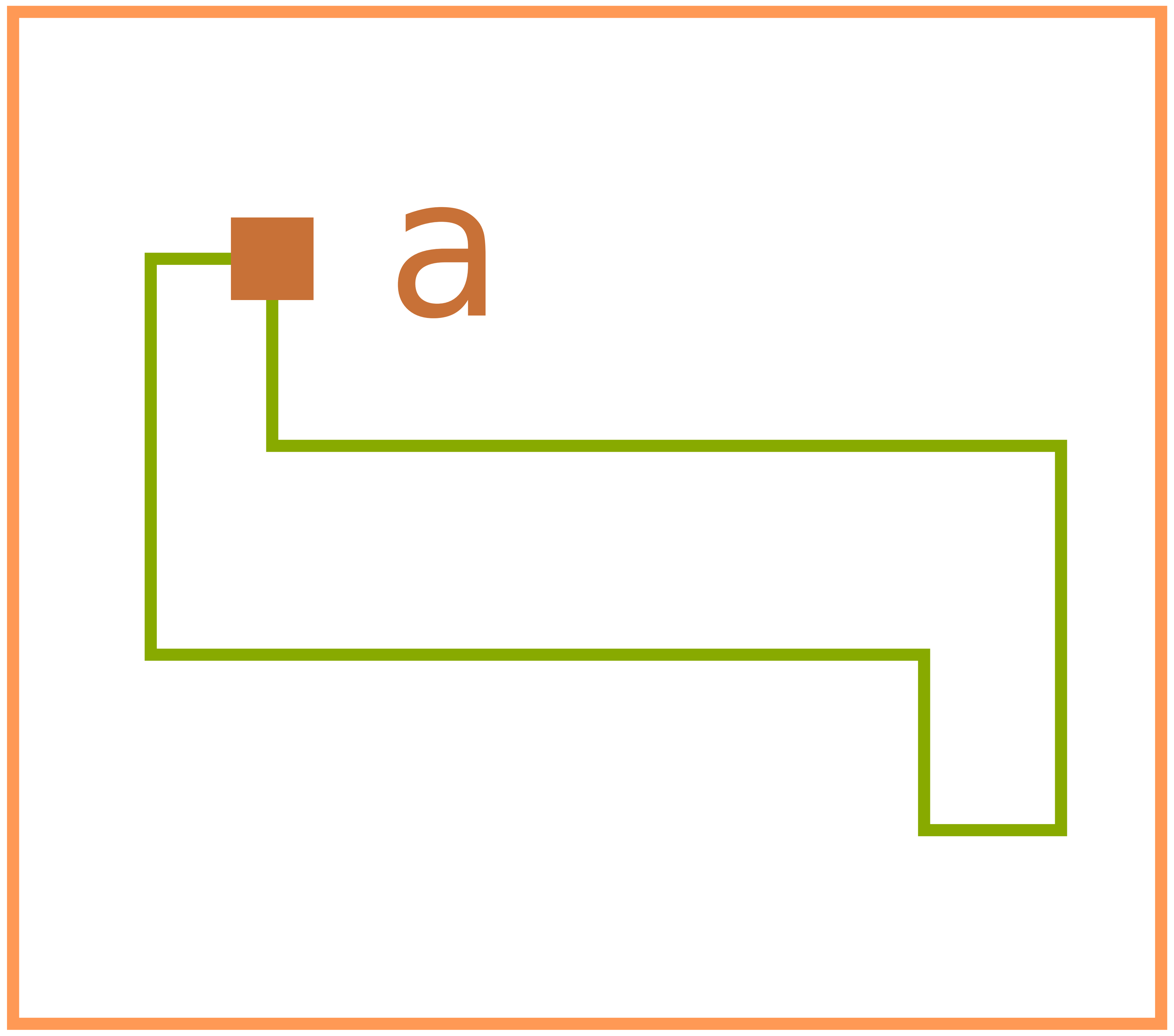
Subject of Study

fundamental groups of pushouts

"structure of loops"

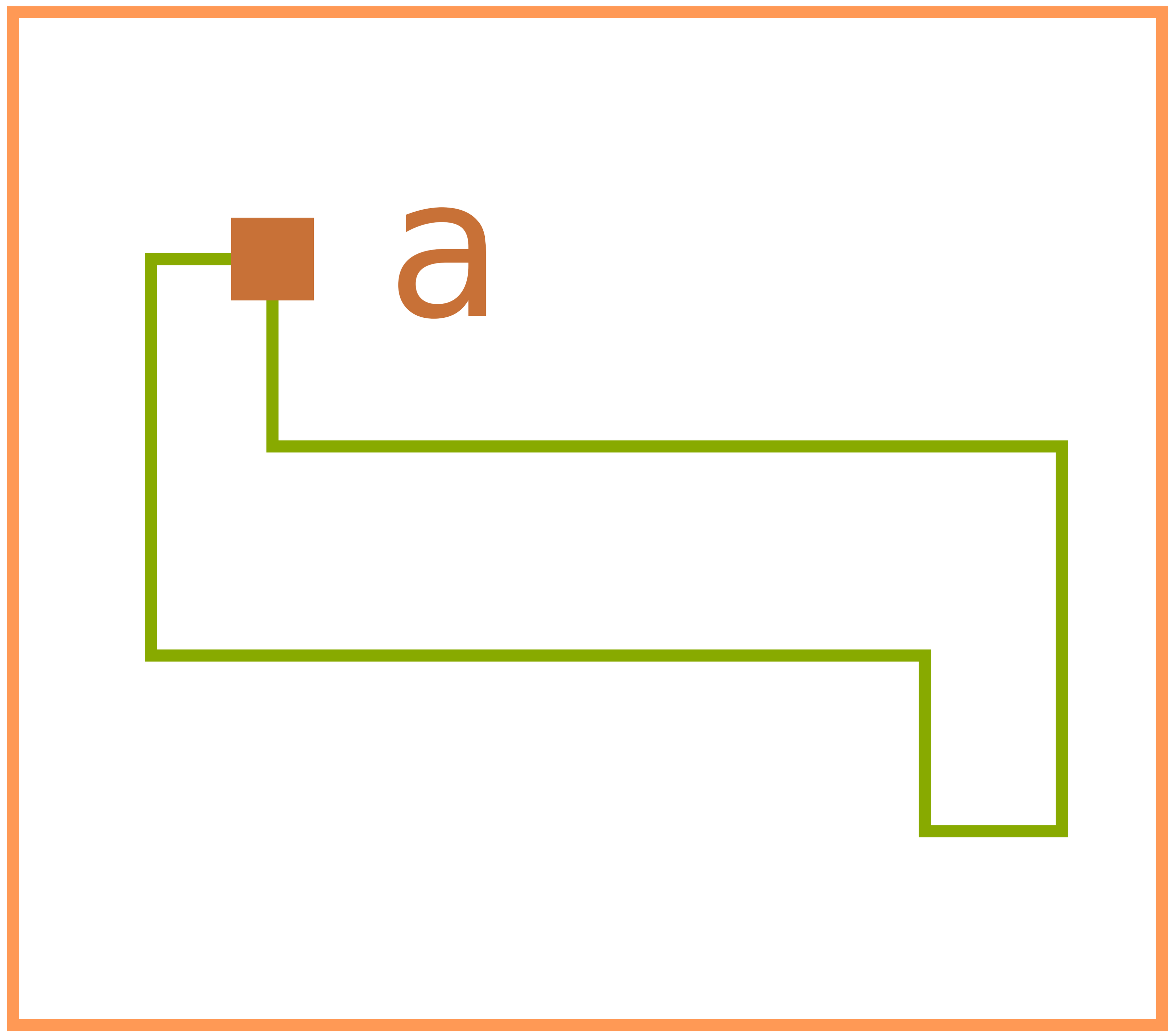
"disjoint union added with bridges"

Fundamental Groups



(unique) ways to travel from a to a

Fundamental Groups

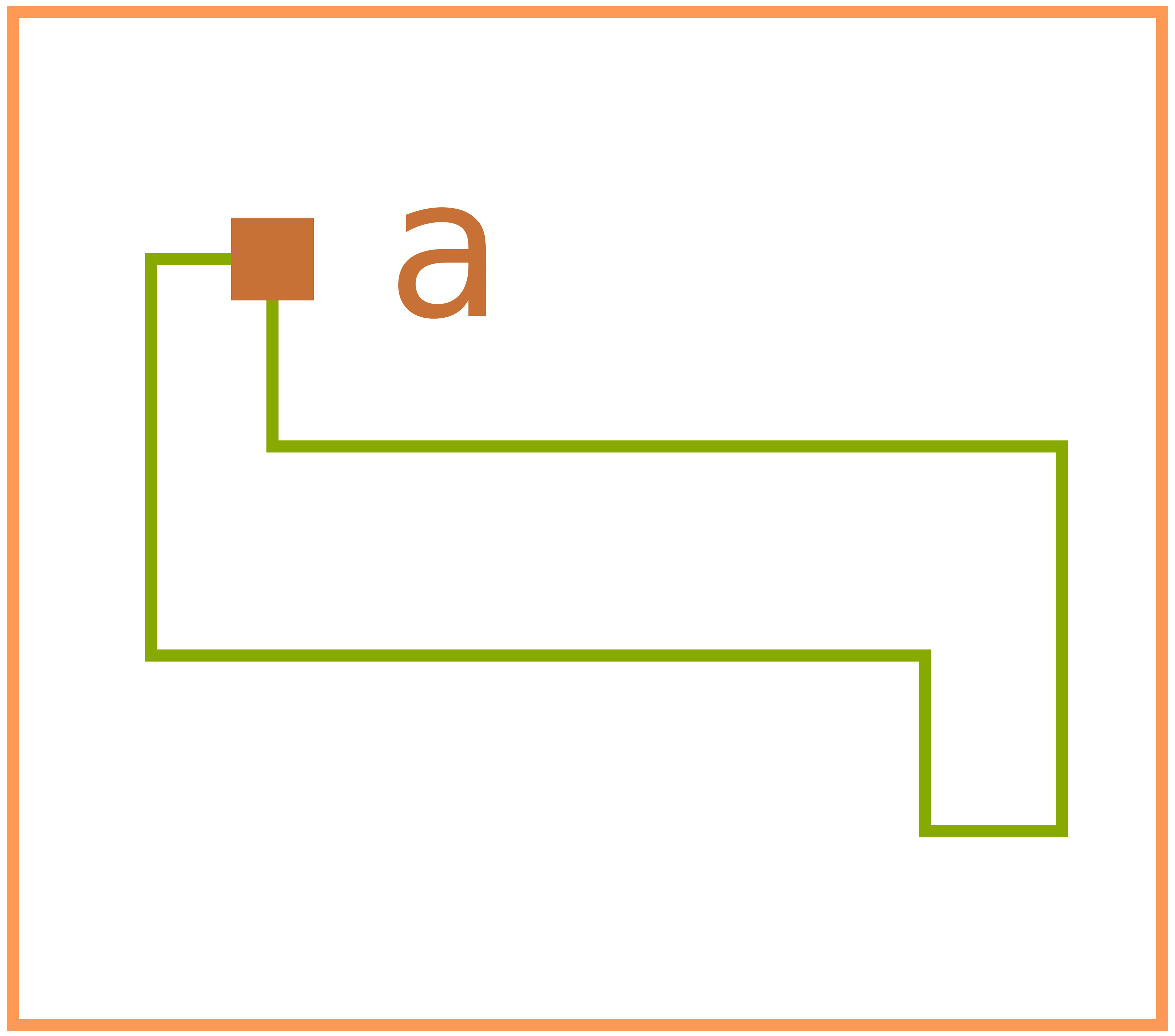


(unique) ways to travel from a to a

here they correspond to integers

positive <--> clockwise
negative <--> counter
zero <--> staying

Fundamental Groups



(unique) ways to travel from a to a

here they correspond to integers

positive <--> clockwise
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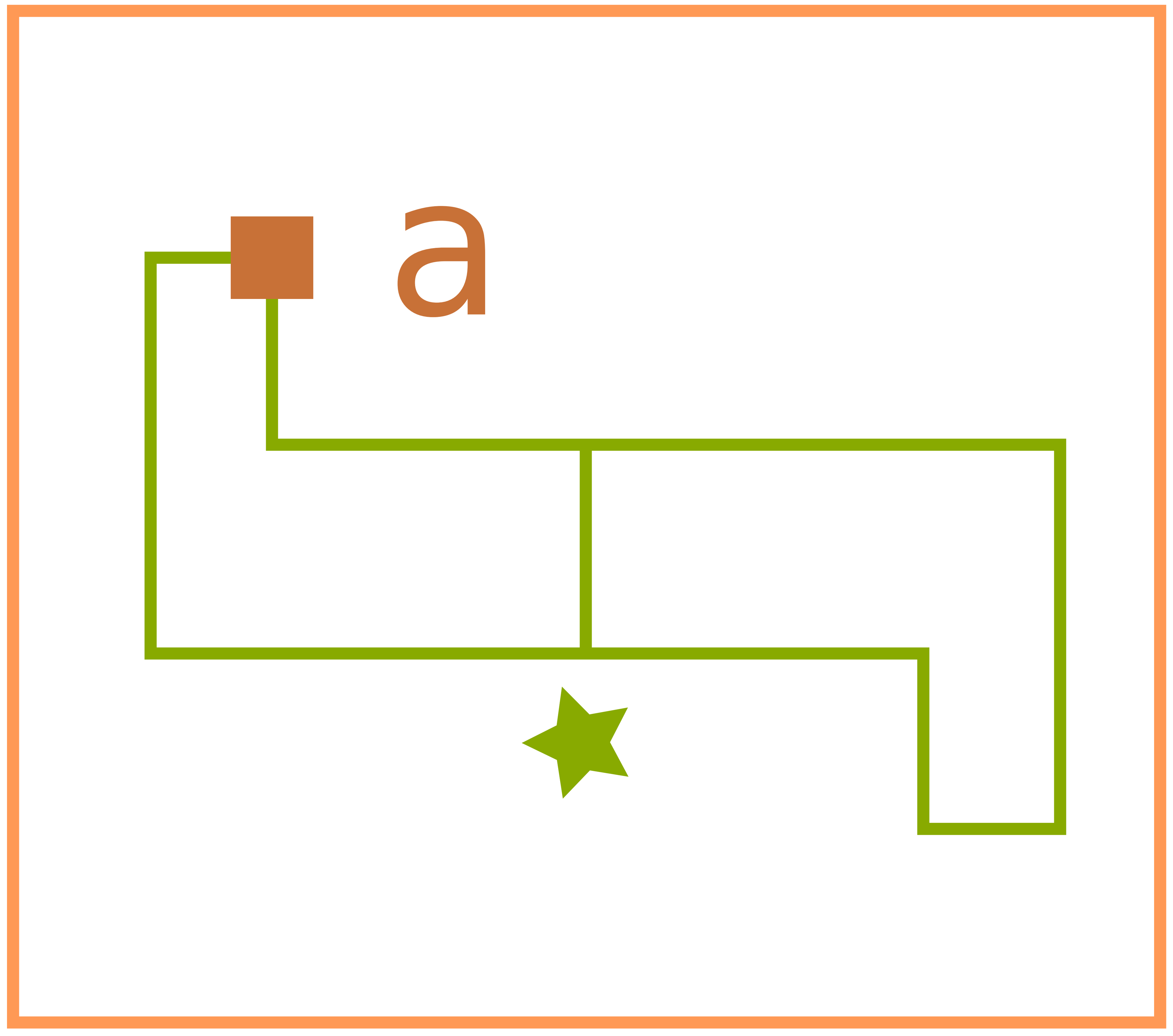
zero <--> staying

Trunc 0 (a == a) ~= Z

Fundamental Groups



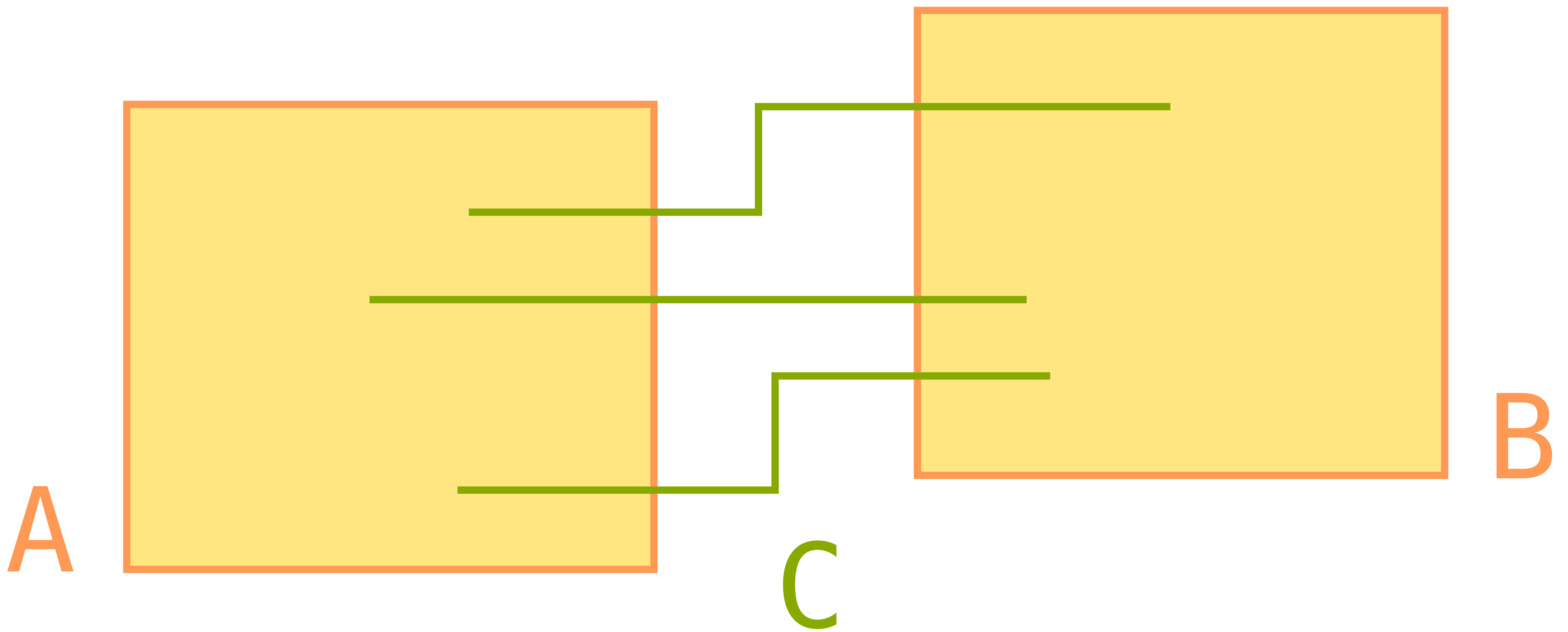
(unique) ways to travel from a to a

much more if a new path ★ is added

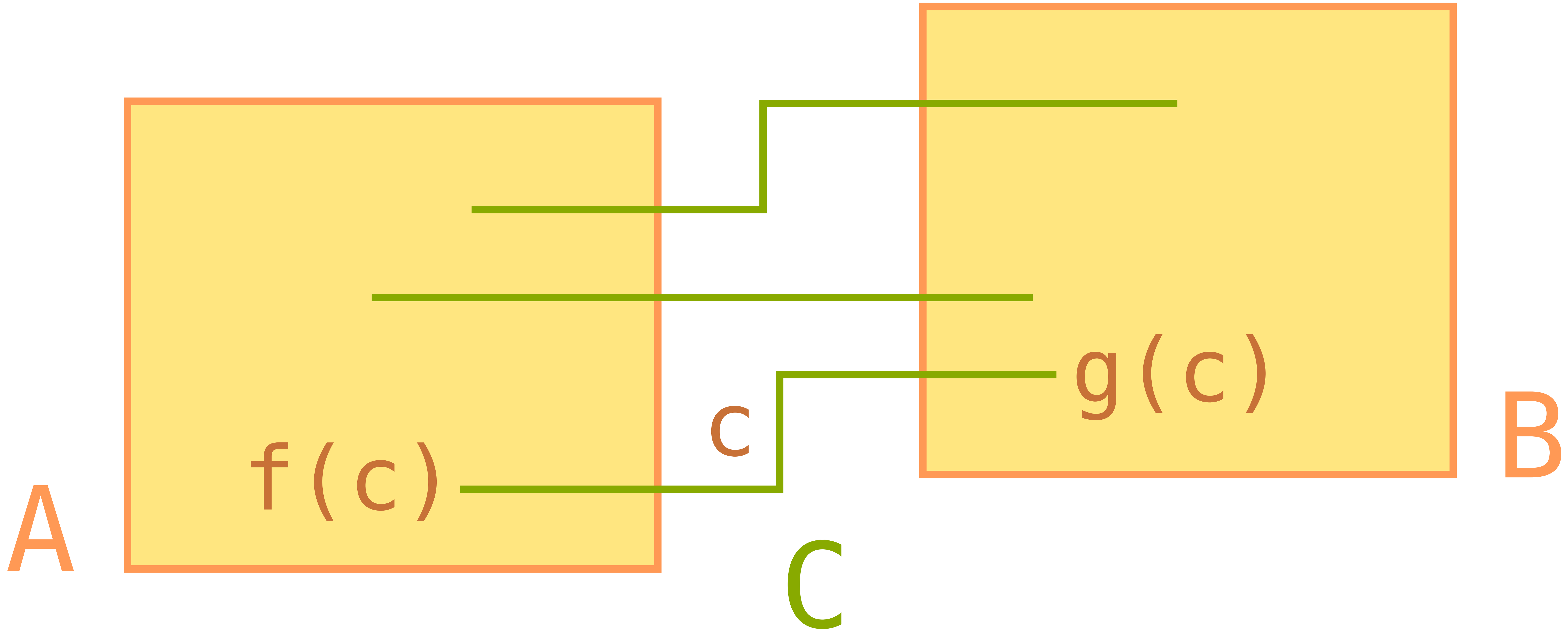
Trunc 0 (a == a) ~= Z * Z

(free product)

(Homotopy) Pushouts

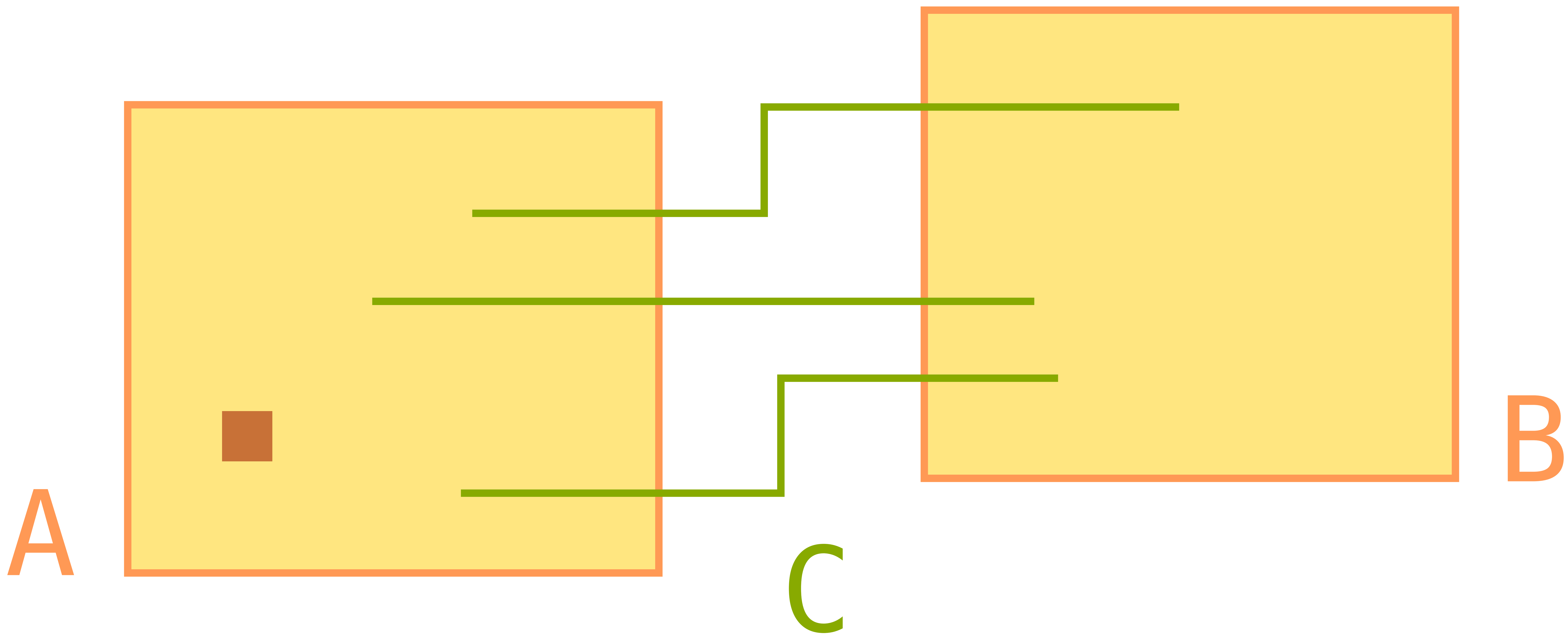


(Homotopy) Pushouts



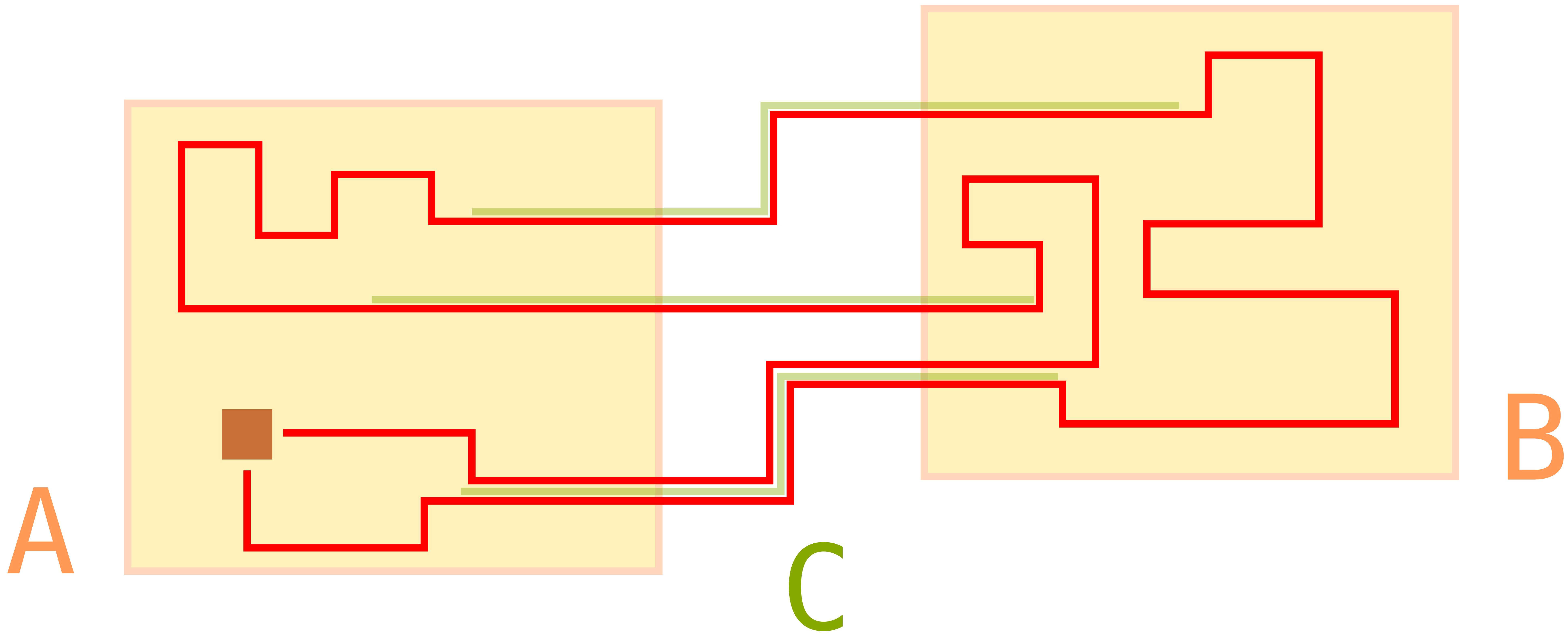

```
data Pushout (A B C : Type)
             (f : C -> A) (g : C -> B) : Type where
  left  : A -> Pushout A B C f g
  right : B -> Pushout A B C f g
  glue  : (c : C) -> left (f c) == right (g c)
```

(Homotopy) Pushouts

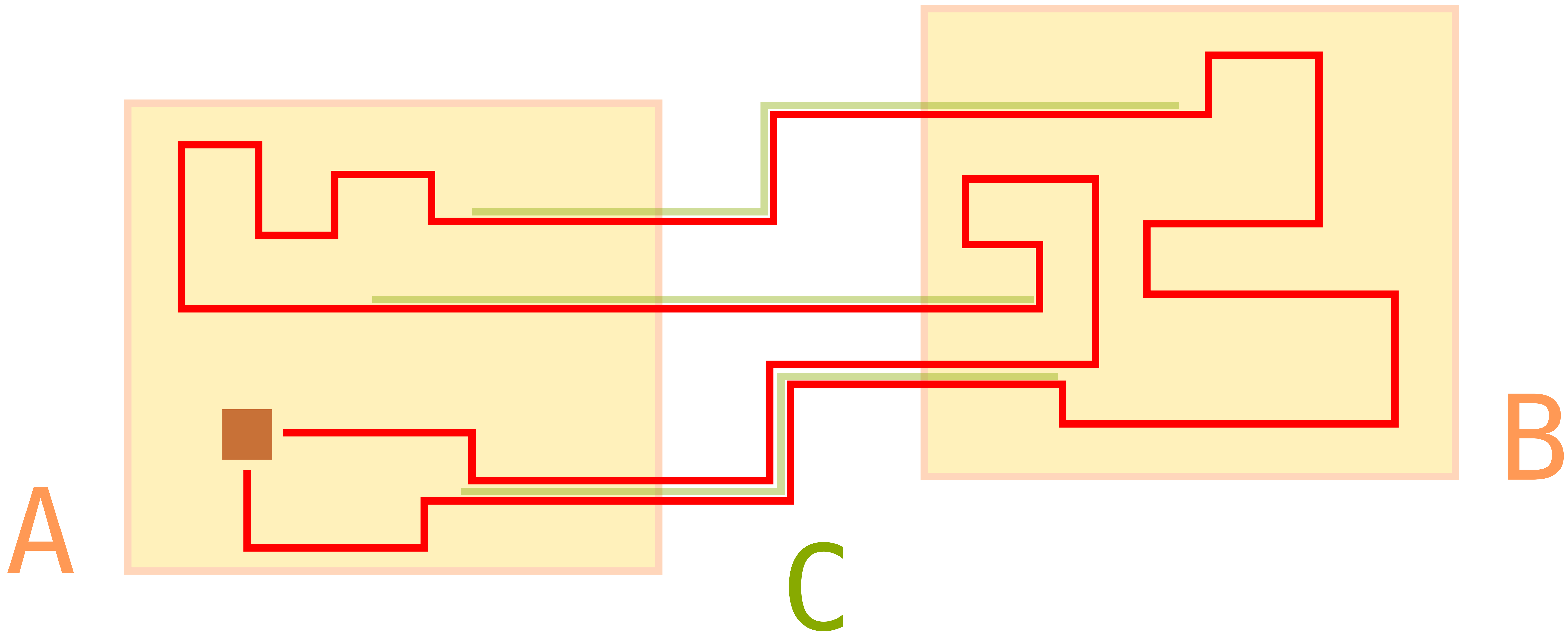


ways to travel from ■ to ■ ?

(Homotopy) Pushouts



(Homotopy) Pushouts



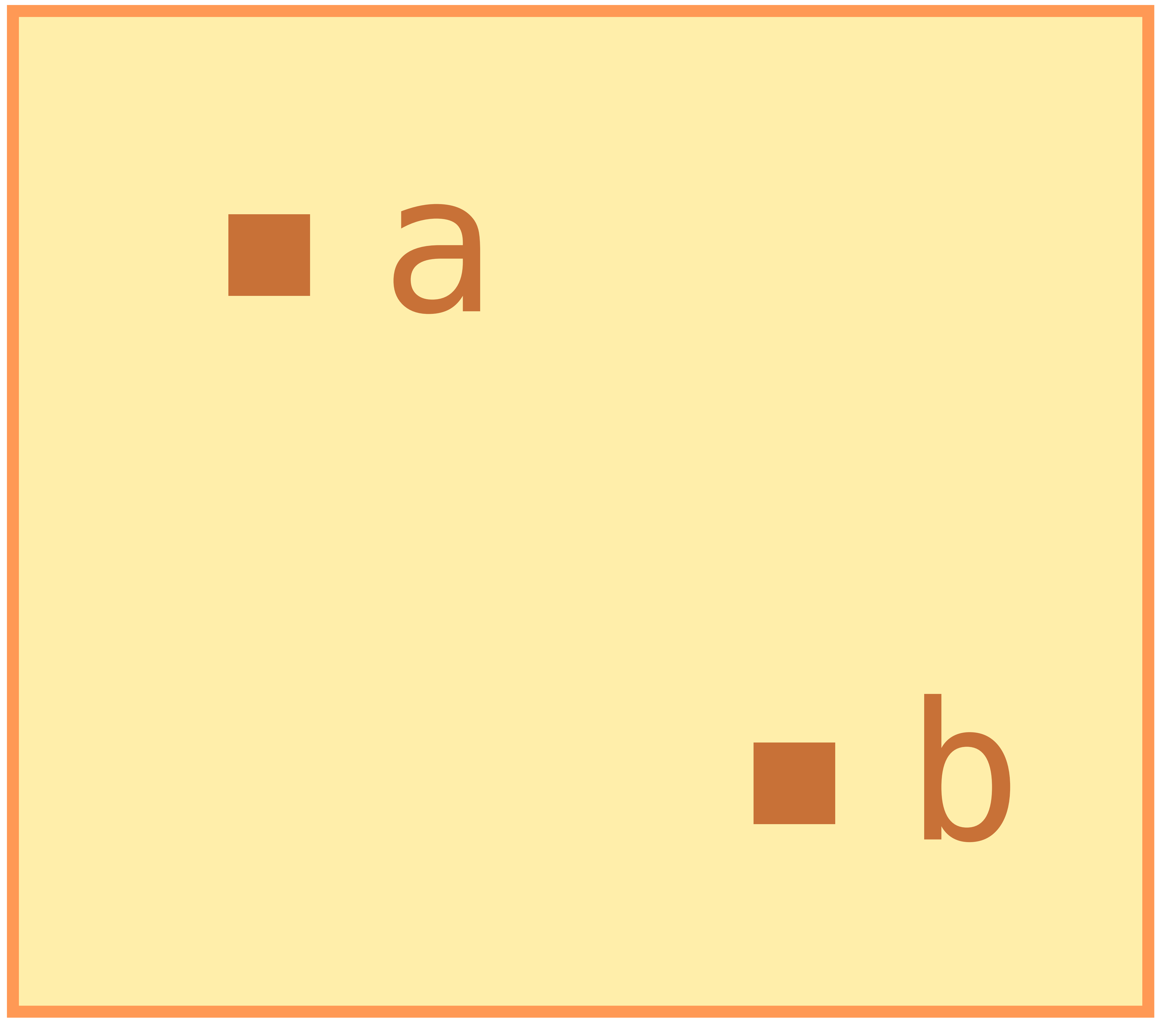
alternative paths in A and B

Theorem Statement

```
for any A, B, C, f and g,
fund.grp(pushout)
        ~= ?(??(A), ??(B), C)
```

??: paths between any two points

Fundamental Groupoids



(unique) ways to travel from a to b

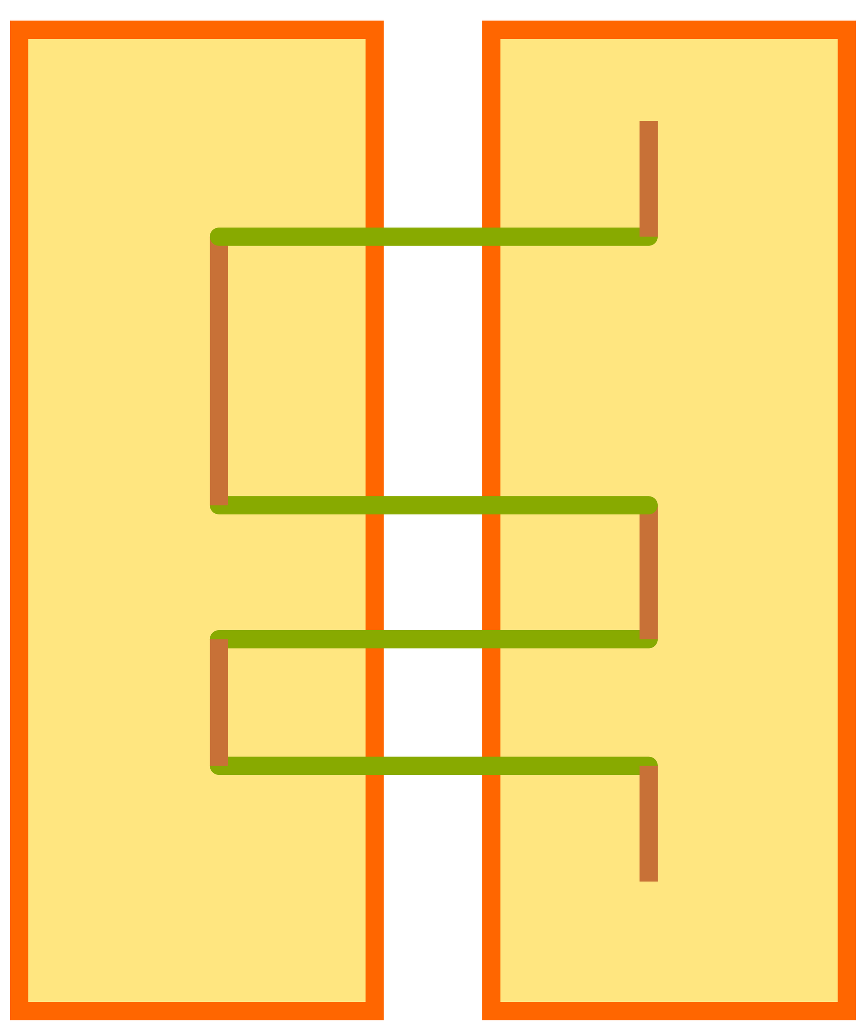
```
Trunc 0 (a == b)
```

Theorem Statement

```
for any A, B, C, f and g,
fund.groupoid(pushout)
      ~= ?(fund.groupoid(A),
           fund.groupoid(B), C)
```

?: "seqs of alternative elems"

Alternative Sequences



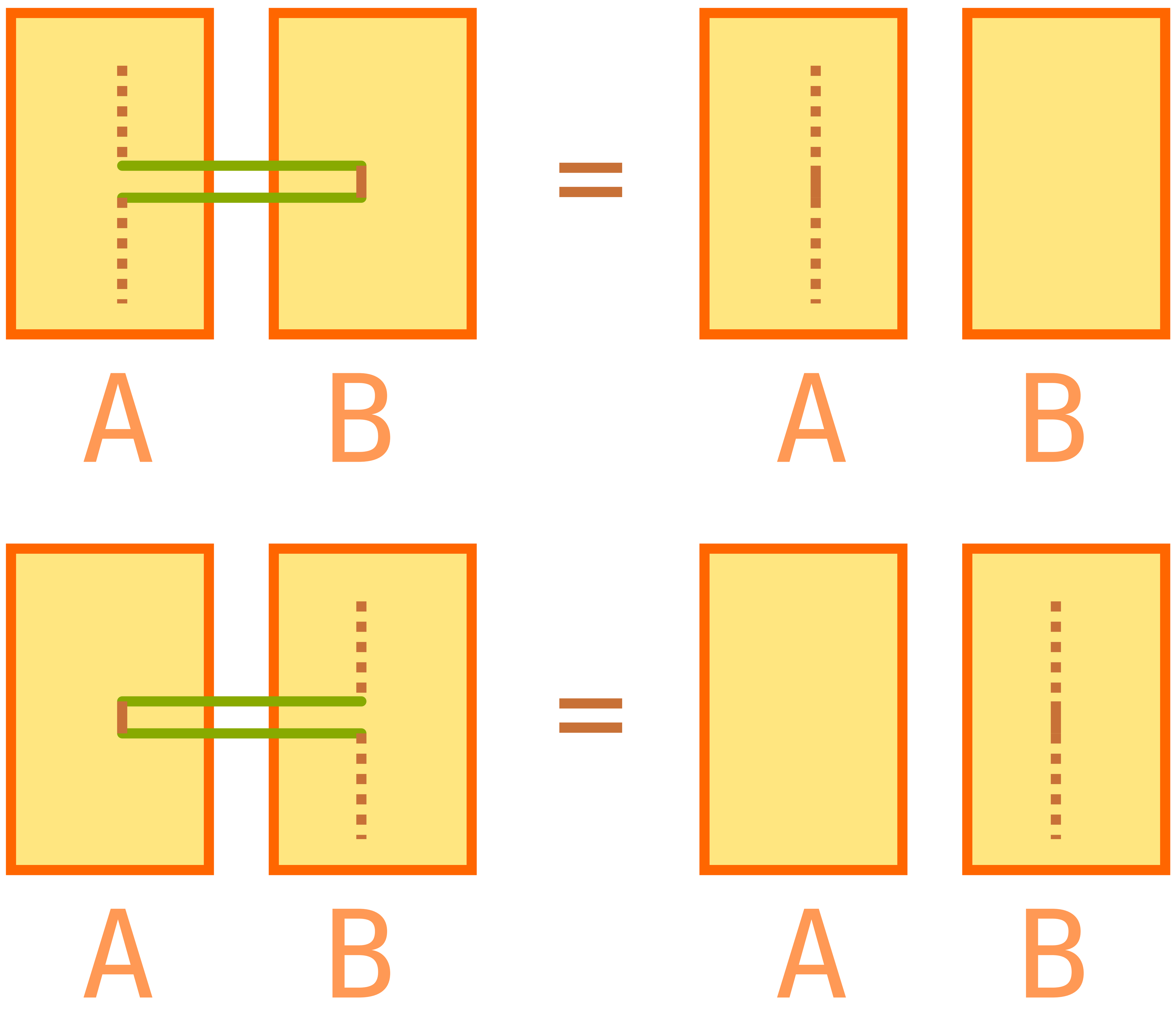
[p1, p2, ..., pn]

induction on both ends:

A to A, A to B,

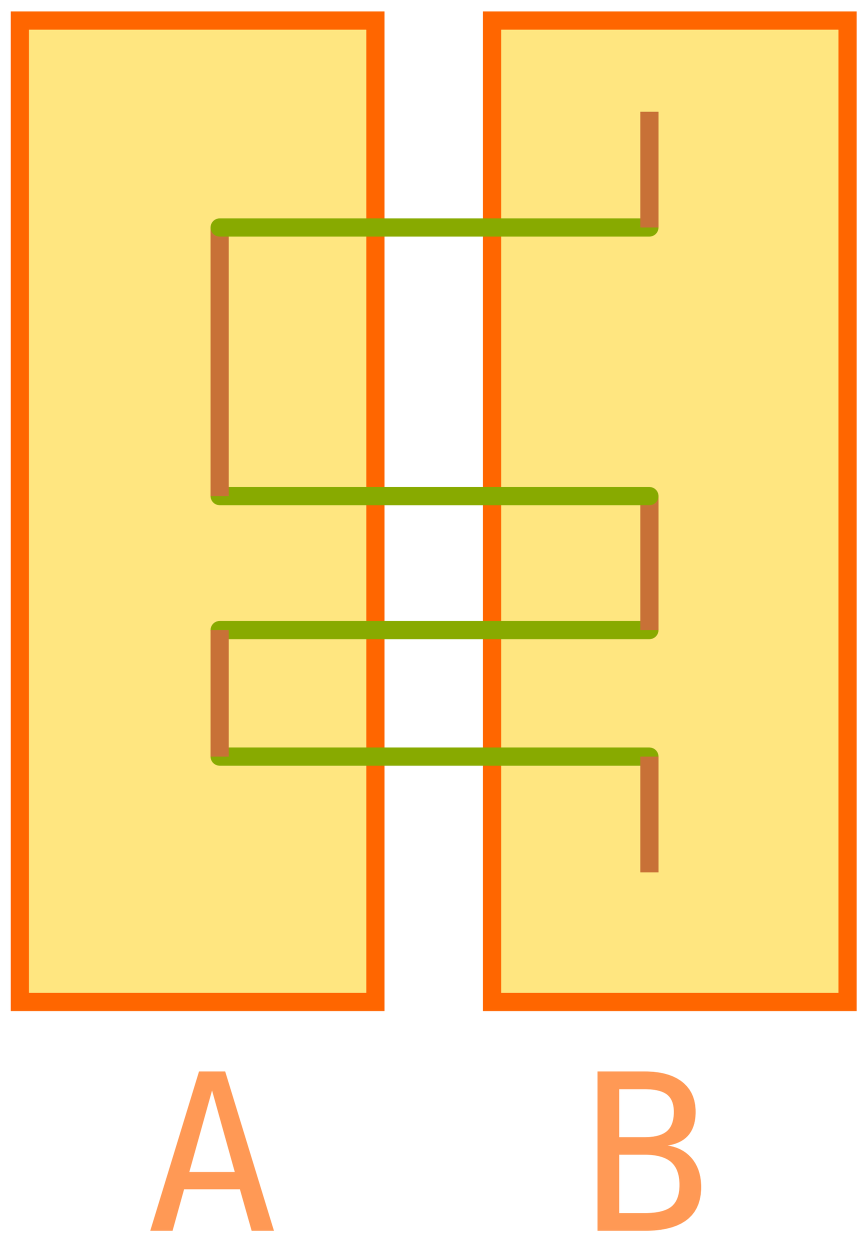
B to A, B to B

Alternative Sequences



quotients of alternative sequences by killing trivial identifications

Alternative Sequences



[p1, p2, ..., pn]

induction on both ends:

A to A, A to B,

B to A, B to B

each case is a quotient
of alternative sequences

Alternative Sequences

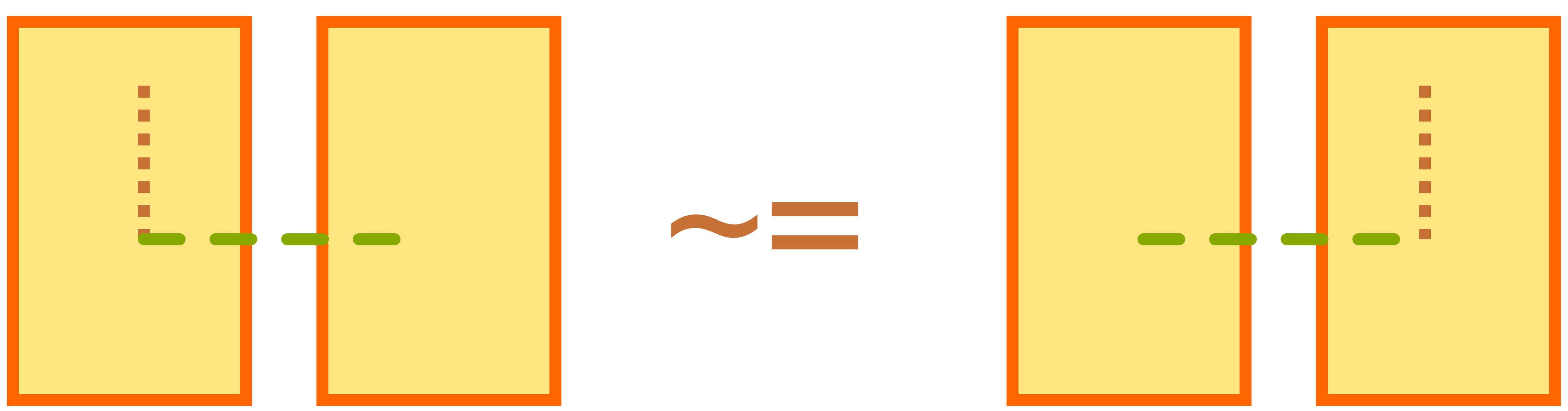
```
next: unify four cases into
      one type family alt.seq
```

Alternative Sequences

next: unify four cases into
one type family alt.seq

show respects for bridges by C. ex:

alt.seq a (f c) ~= alt.seq a (g c)



```
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```

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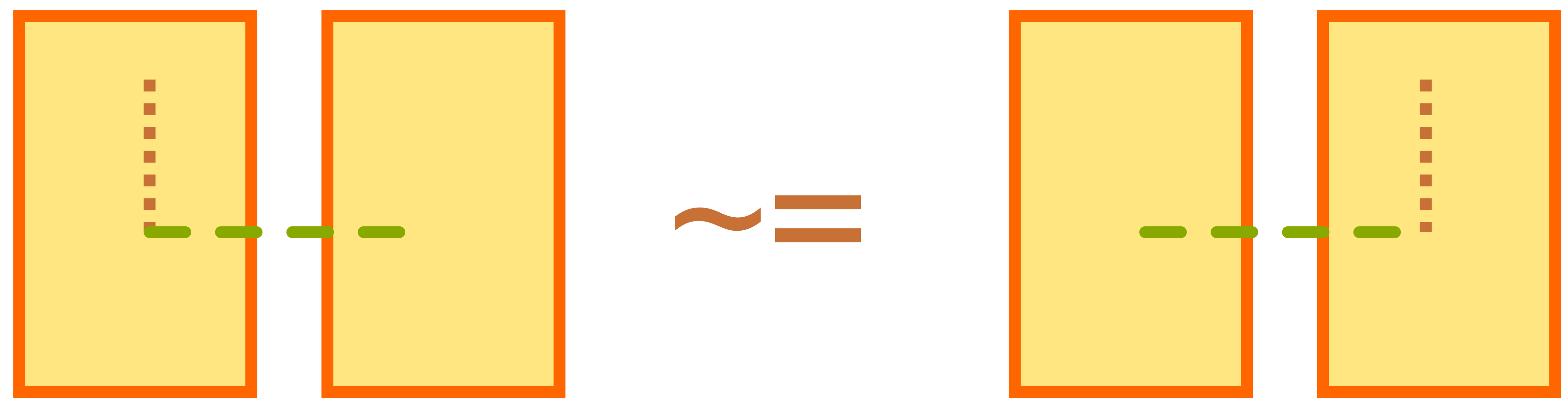
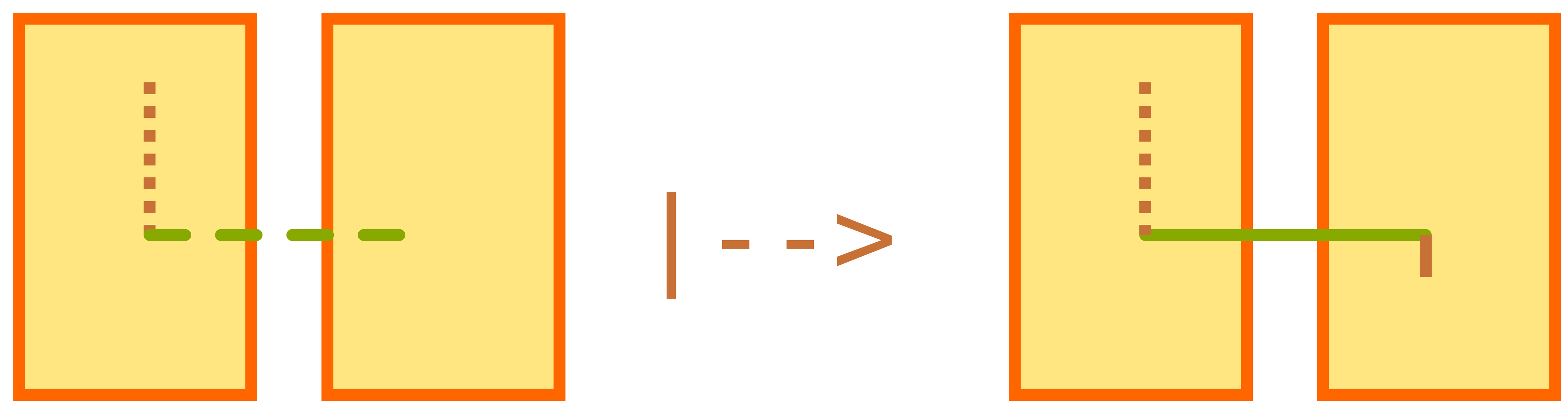
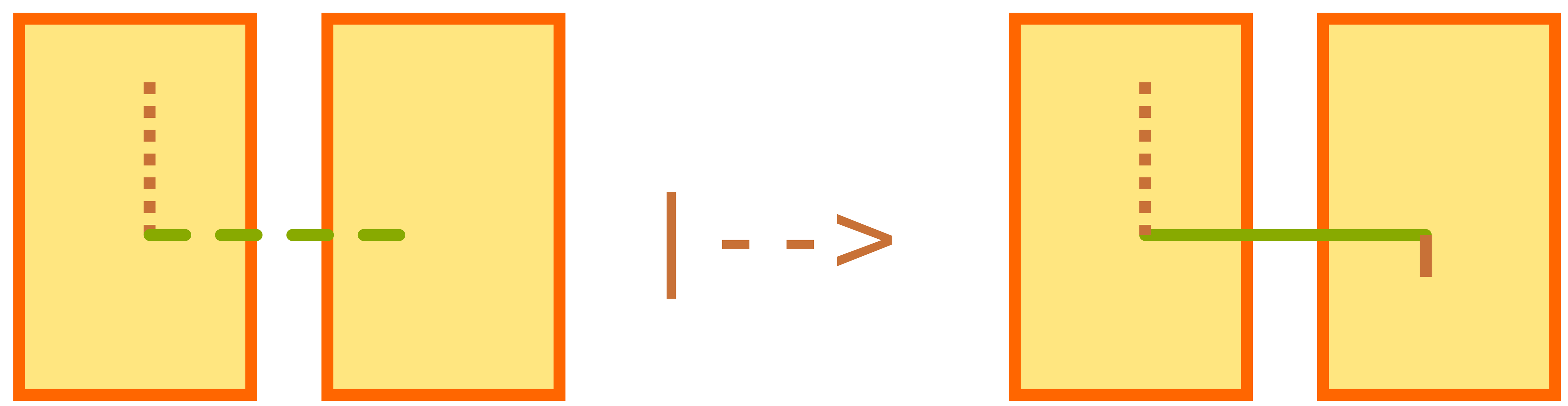


[..., p] |--> [..., p, trivial]



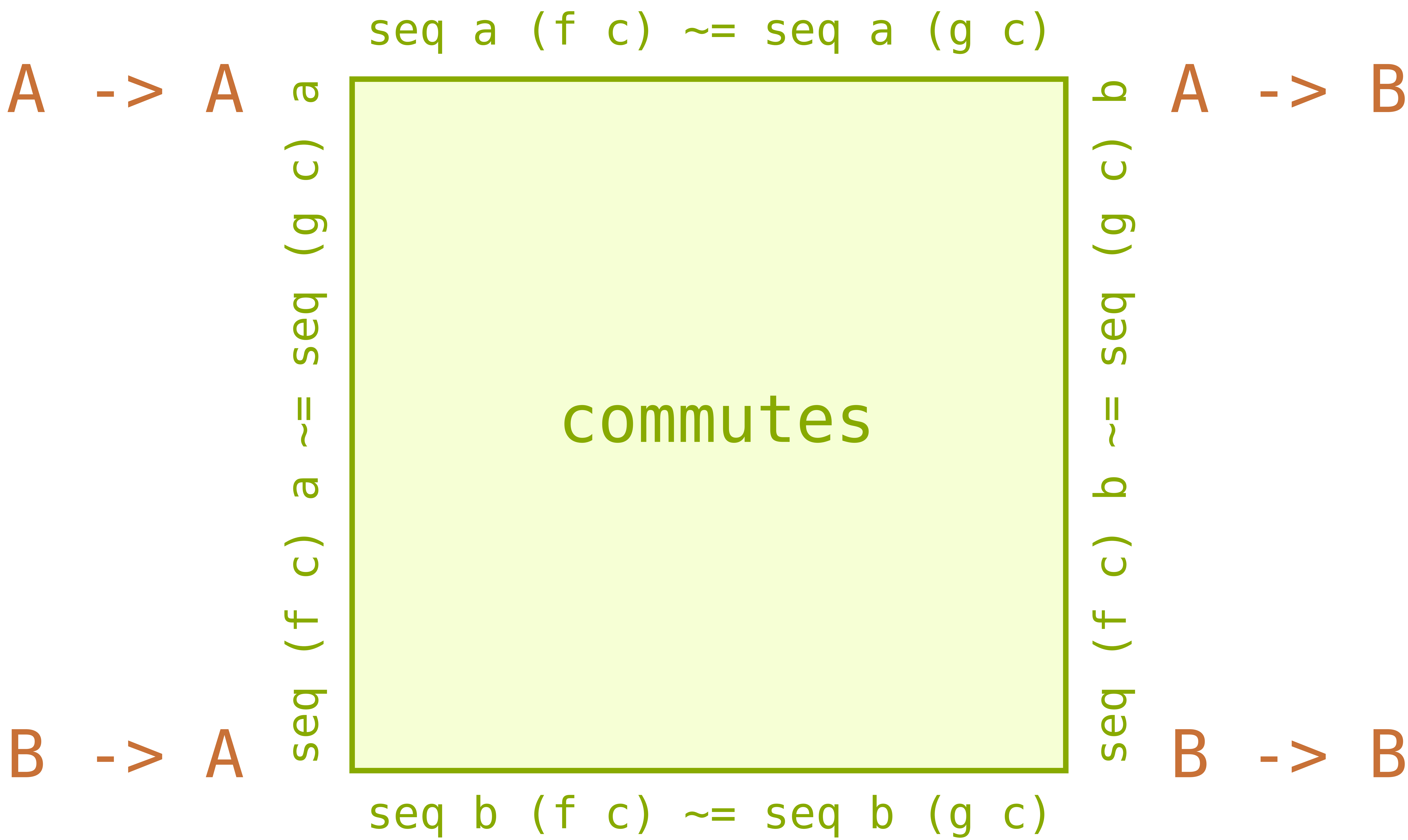
alt.seq a (f c) ~= alt.seq a (g c)

[..., p] |--> [..., p, trivial]



[..., p, trivial] <--| [..., p]

Alternative Sequences



Theorem

```
for any A, B, C, f and g,
fund.groupoid(pushout)
 ~= alt.seqs(fund.groupoid(A),
             fund.groupoid(B), C)
```

(zero pages left before the proofs)

Recipe of Equivalences

* two functions back and forth
  ("decode" and "encode")

* round-trips are identity

```
fund.groupoid -> alt.seqs
                encode
    (all paths)

 Trunc 0 (p == q) -> alt.seqs p q
```

```
fund.groupoid ->encode alt.seqs
    (all paths)

 Trunc 0 (p == q) -> alt.seqs p q

   path induction:
     consider only trivial paths

 (p : Pushout) -> alt.seqs p p
```

fund.groupoid ->encode alt.seqs

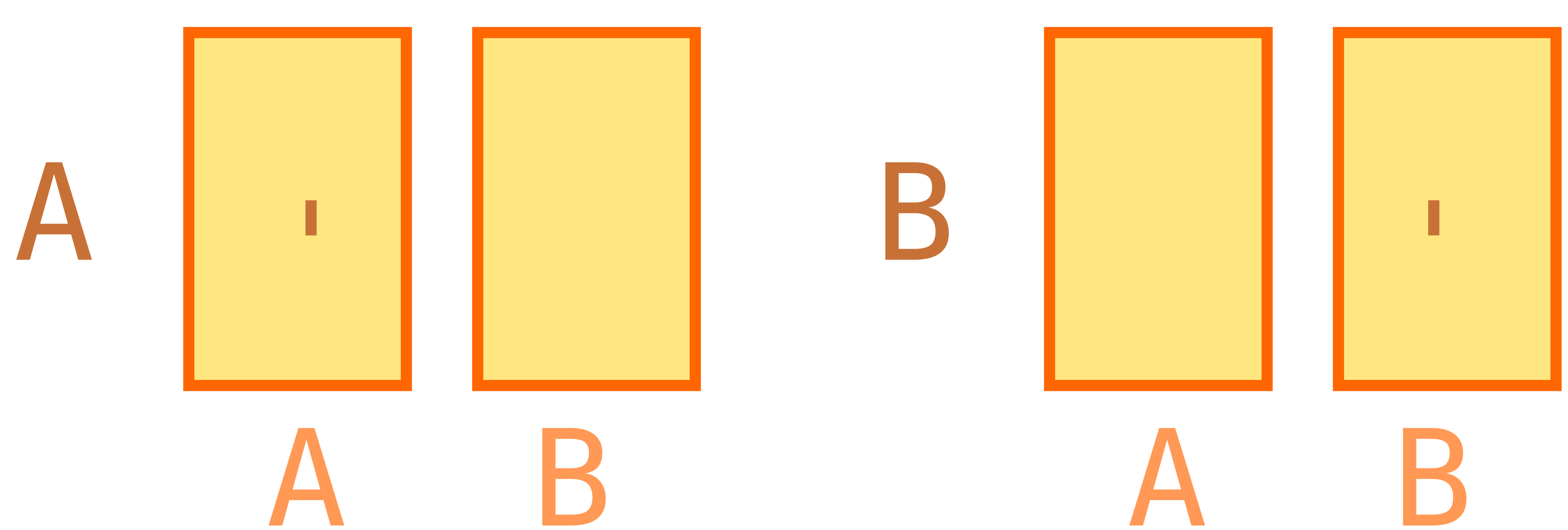
(all paths)

Trunc 0 (p == q) -> alt.seqs p q

path induction:
consider only trivial paths

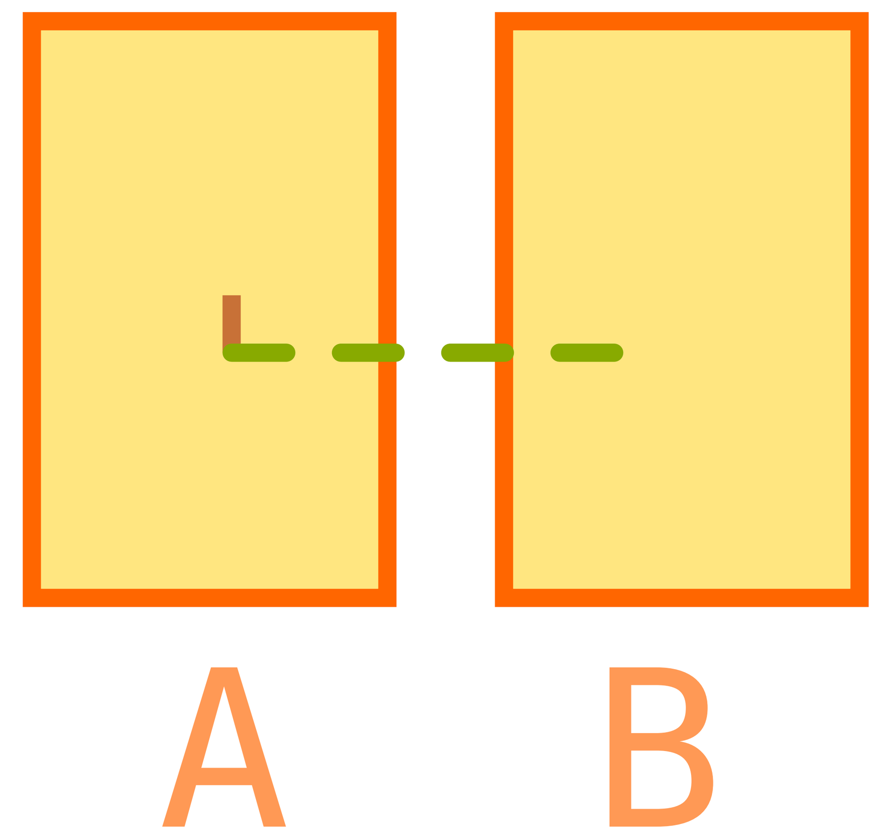
(p : Pushout) -> alt.seqs p p

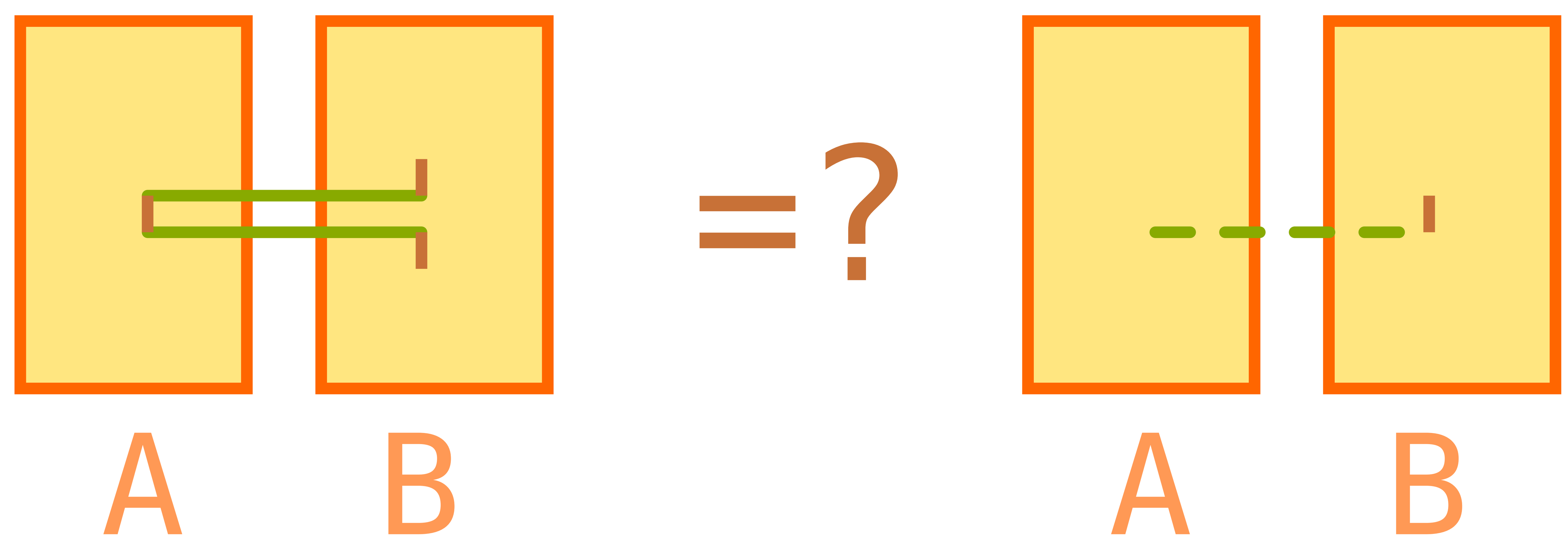
pushout induction



bridges by C
(next page)

case A

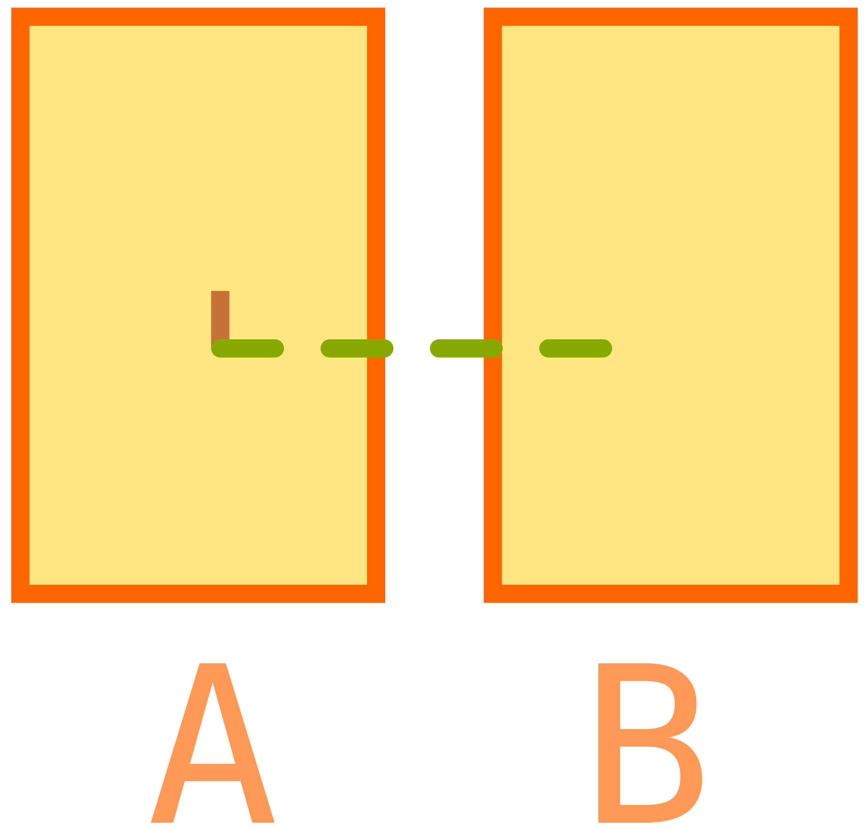


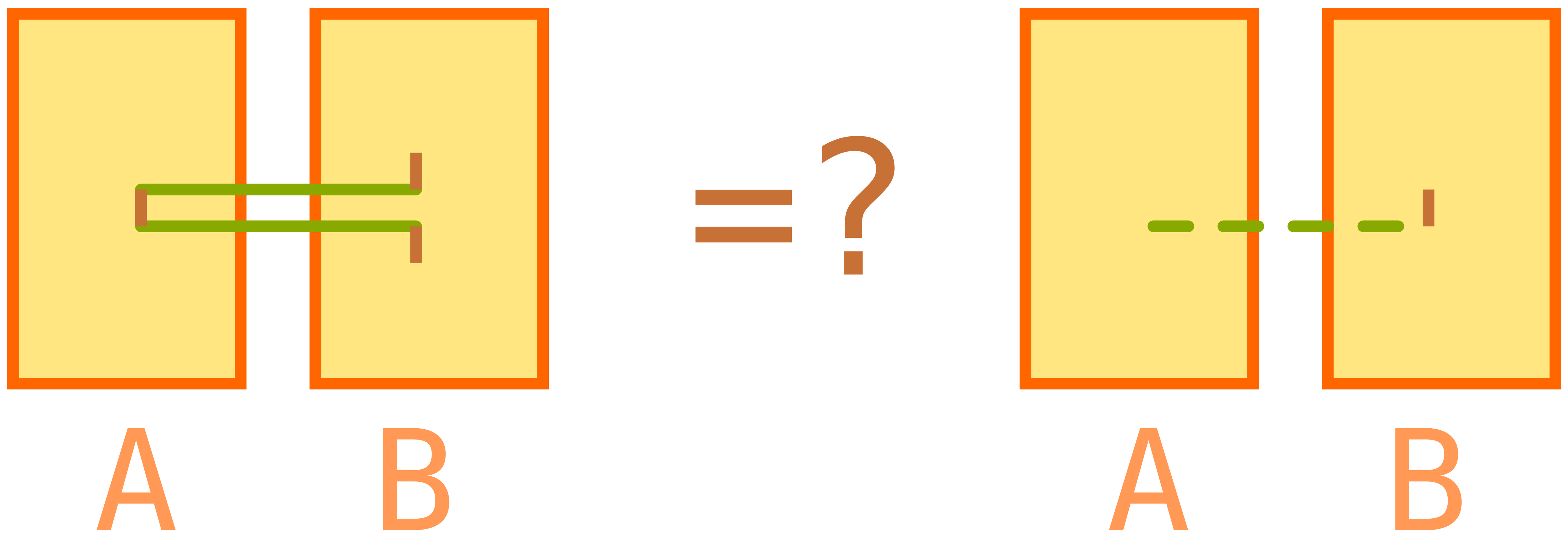


case B

applying the diagonal
in coherence square

case A





case B

applying the diagonal
in coherence square

witnessed by the quotient

alt.seq -> fund.groupoid

decode

just compositions!

$\text{alt.seq} \xrightarrow[\text{decode}]{} \text{fund.groupoid}$

just compositions!

$\text{grpd} \xrightarrow[\text{encode}]{} \text{seqs} \xrightarrow[\text{decode}]{} \text{grpd}$

again by path induction
(similar to "encode")

alt.seq $\to_{\text{decode}}$ fund.groupoid

just compositions!

grpd $\to_{\text{encode}}$ seqs $\to_{\text{decode}}$ grpd

again by path induction
(similar to "encode")

seqs $\to_{\text{decode}}$ grpd $\to_{\text{encode}}$ seqs

induction on sequences

```
lemma: encode(decode[p1,p2,...])
     = p1 :: encode(decode[p2,...])
```

Theorem

```
for any A, B, C, f and g,
fund.groupoid(pushout)
  = alt.seqs(fund.groupoid(A),
             fund.groupoid(B), C)
```

Final Notes

* Refined version: Can focus on just the set of base points of C covering its components.

* All mechanized in Agda
  github.com/HoTT/HoTT-Agda/blob/1.0/Homotopy/VanKampen.agda

* Submitted to CSL 2016
  www.cs.cmu.edu/~kuenbanh/files/vankampen.pdf