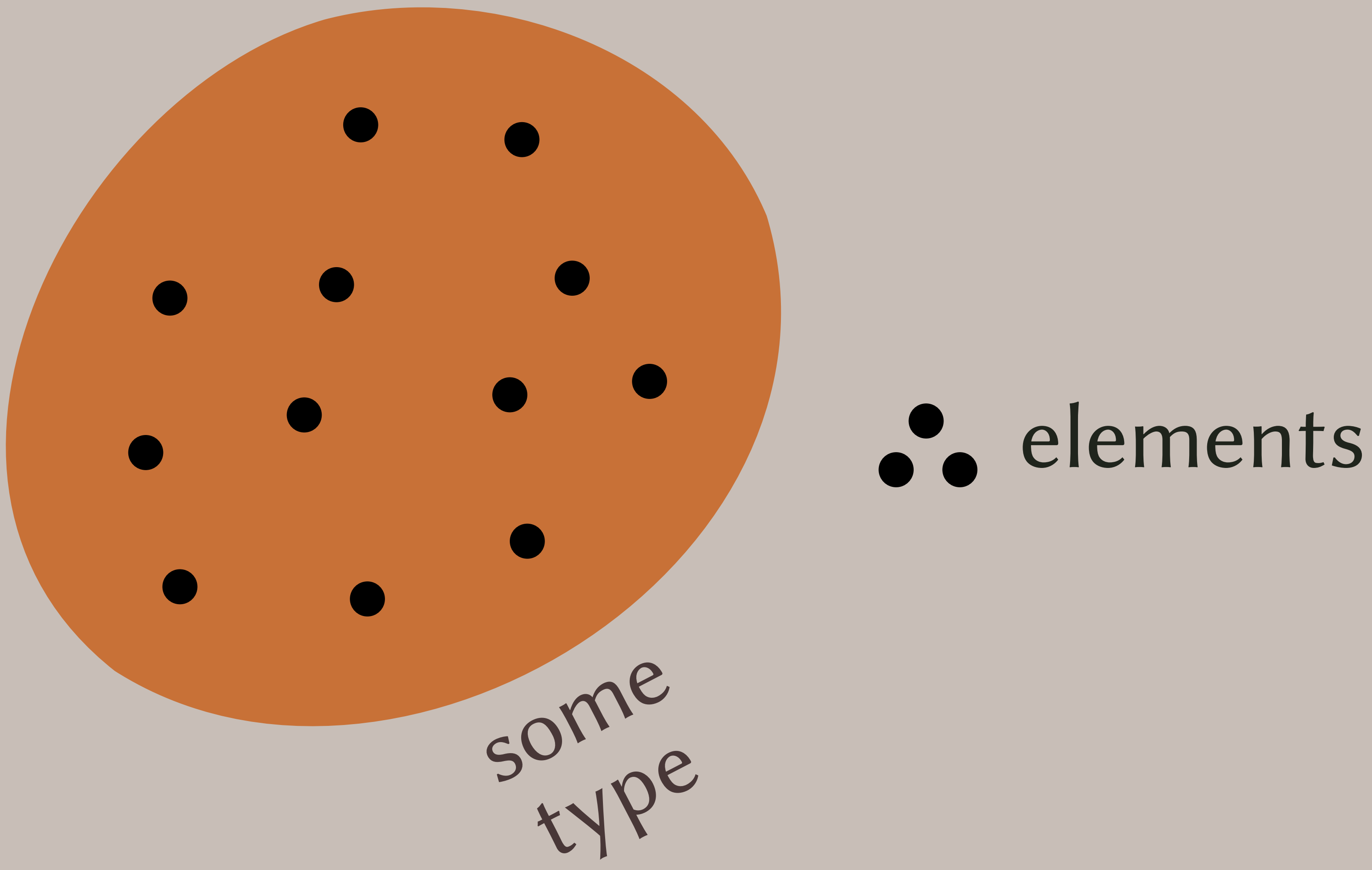
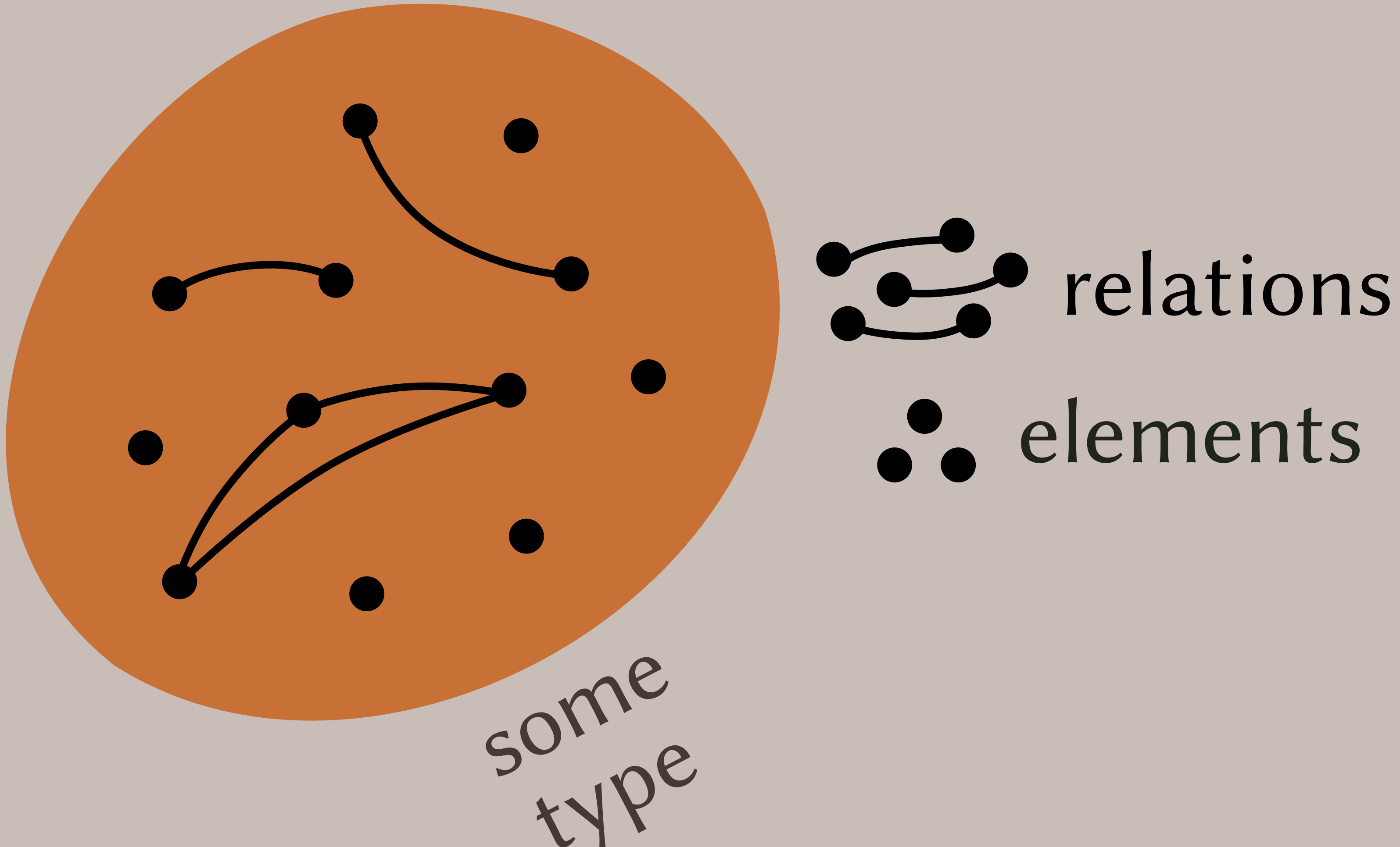
Rise of higher-dimensional TYPES

with director's comments!

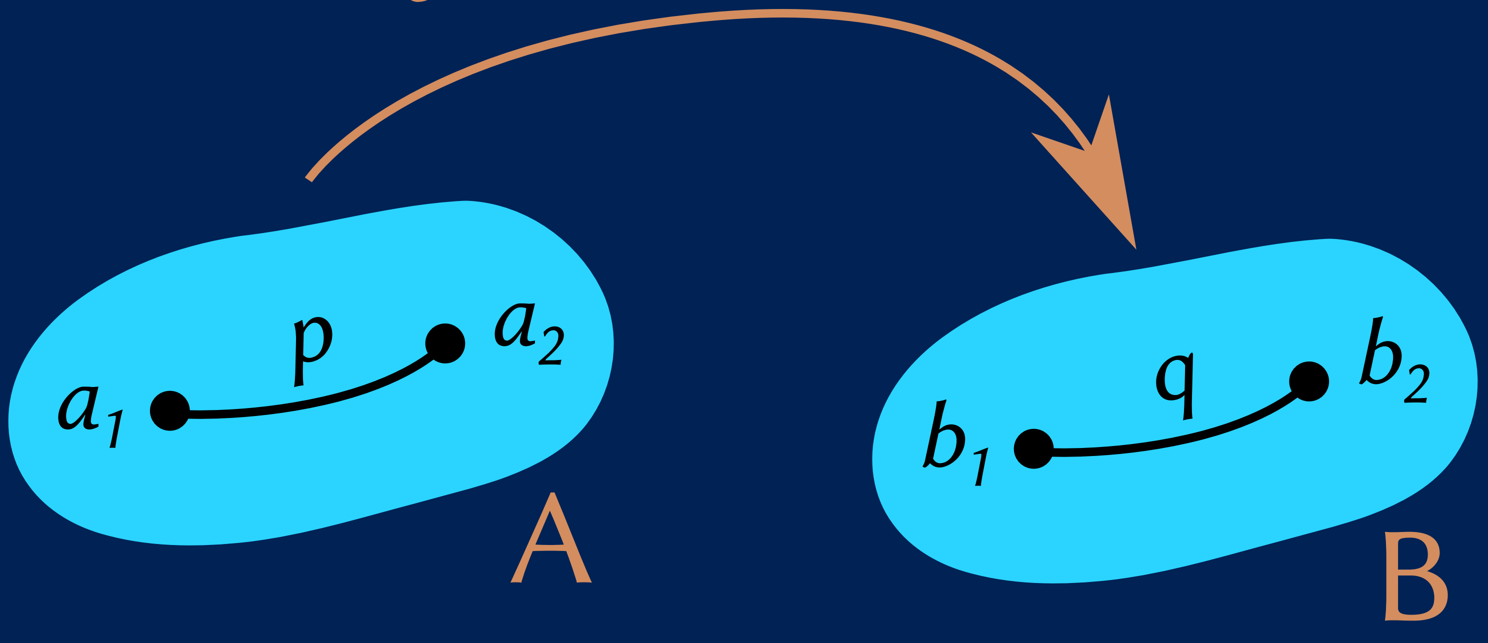
FAVONIA
University of Minnesota


elements
some
type


relations
elements
some
type

$$a_1 \longmapsto b_1$$
$$a_2 \longmapsto b_2$$
$$p \longmapsto q$$

$$f : A \longrightarrow B$$



Higher-dimensional types
provide novel abstraction that facilitates
the mechanization
of homotopy theory

the abstraction

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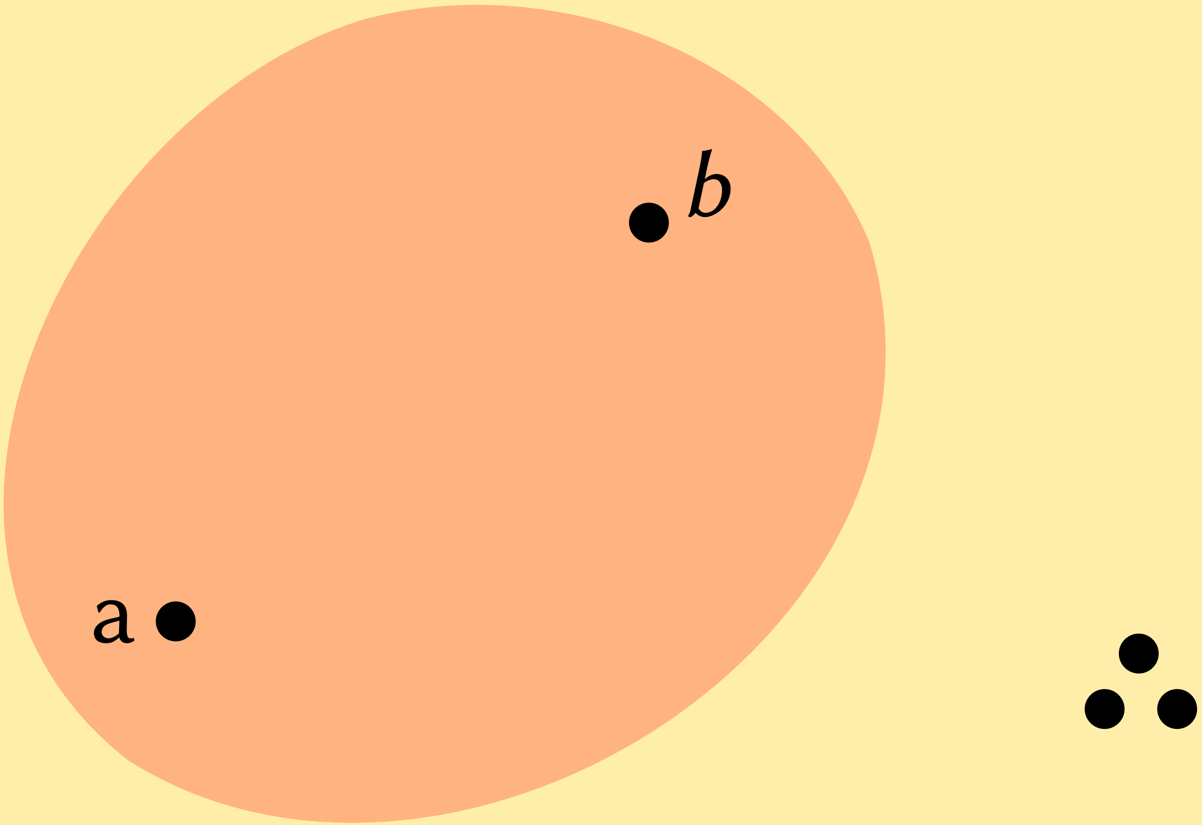
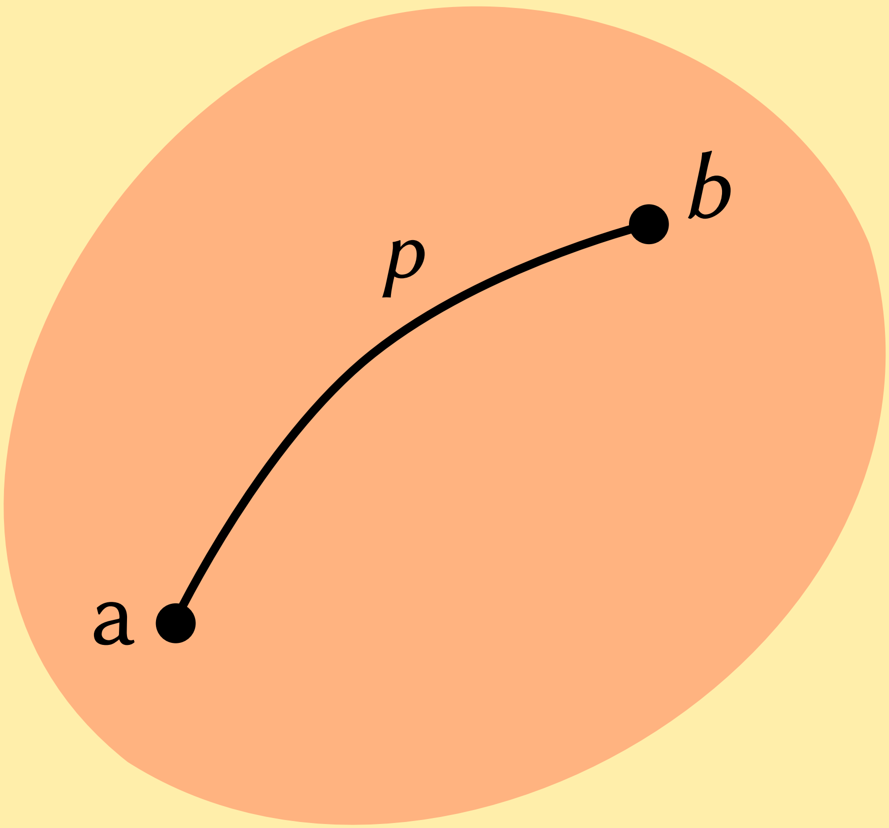
the work

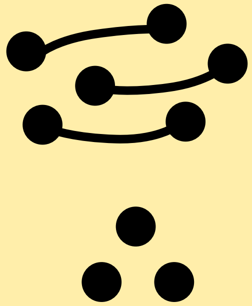
Higher-Dimensional Types
(symmetric relations)



Higher-Dimensional Types

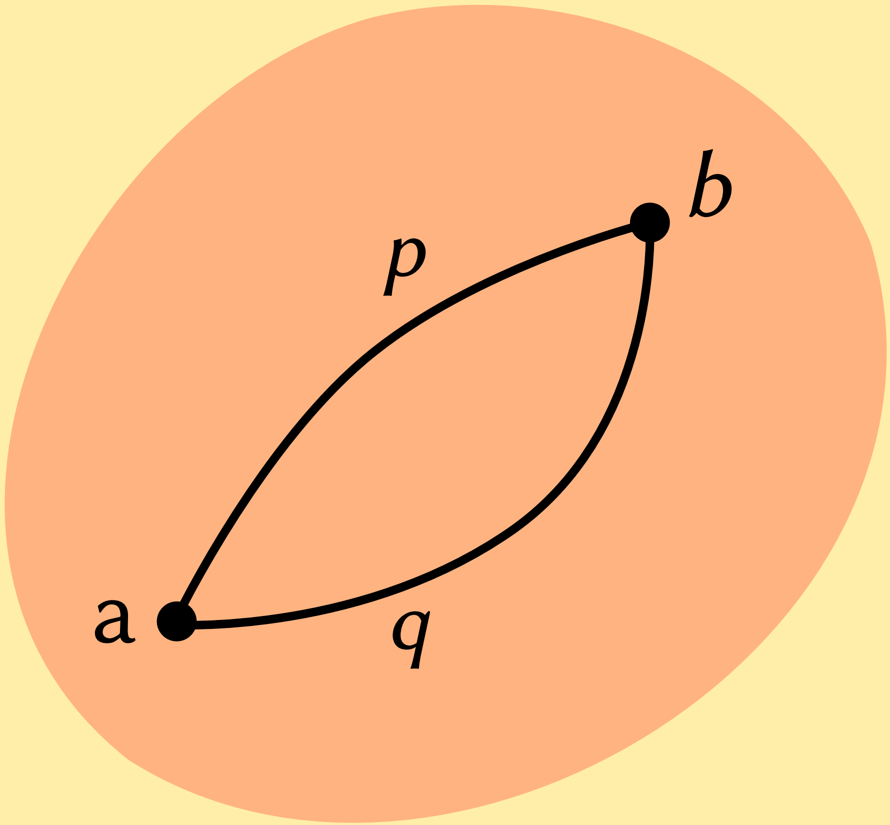
(symmetric relations)

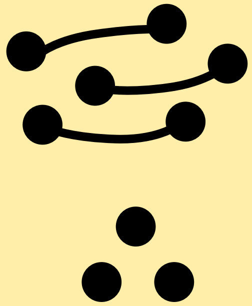




Higher-Dimensional Types

(symmetric relations)





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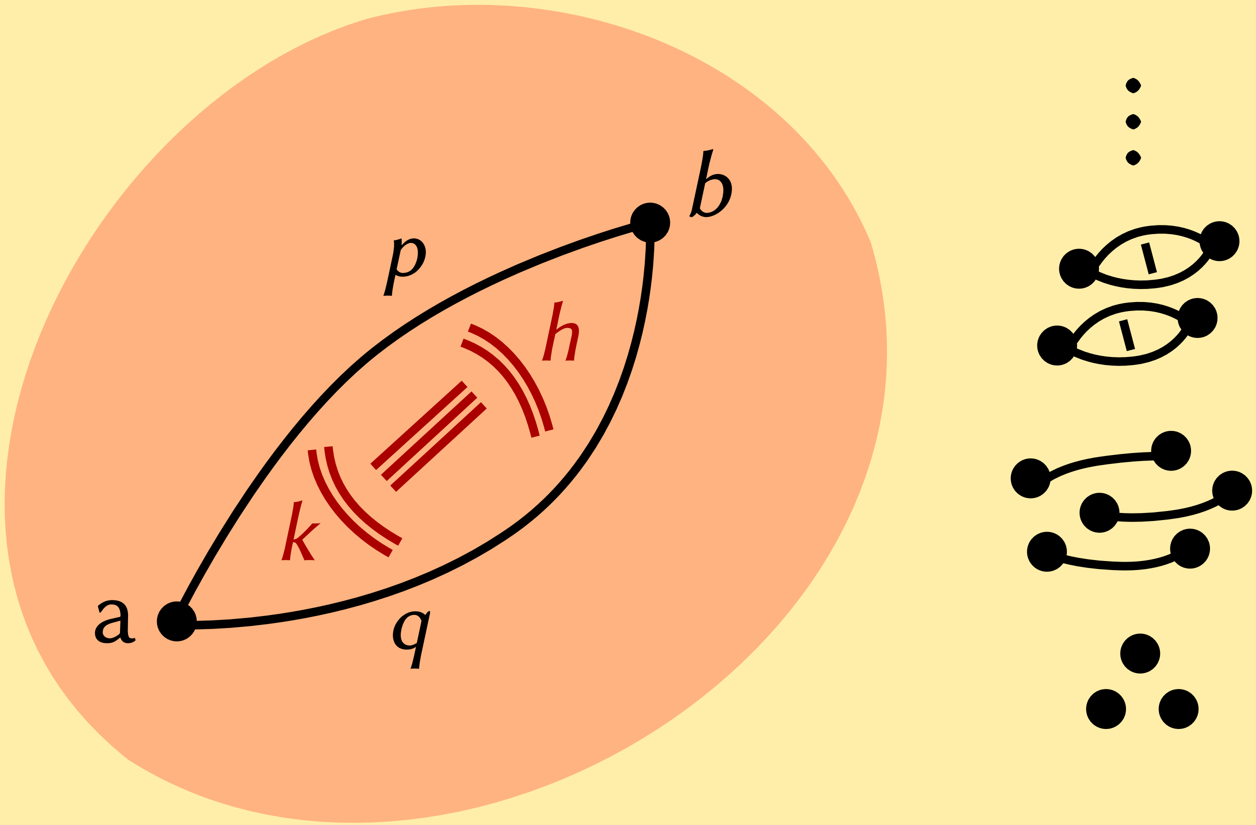


Homotopy-Theoretic Interpretation

[Awodey and Warren] [Voevodsky *et al*]
[van den Berg and Garner]

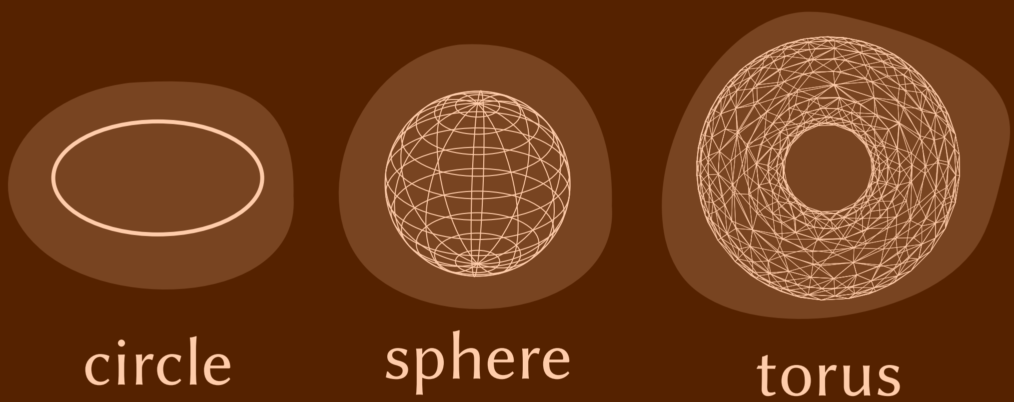
A	Type	Space
$a : A$	Element	Point
$f : A \longrightarrow B$	Function	Continuous Mapping
$C : A \longrightarrow Type$	Dependent Type	Fibration
$a =_A b$	Identification	Path

New Features

Univalence

if e is an equivalence between types A and B, then $ua(e):A=B$

Higher Inductive Types



[STATEMENT]

Higher-dimensional types

provide novel abstraction for mechanization

↑

[EXPERIMENTS]

Mechanizing theorems
in **univalent type theory**

My Thesis + Follow-Ups

Homotopy groups

Covering spaces [Favonia and Harper]

Seifert-van Kampen theorem [Shulman and Favonia]

Blakers-Massey theorem [Lumsdaine, Finster, Licata, Brunerie and Favonia]

Cohomology groups

[Buchholtz and Favonia]

4.2 Elementary Methods of Calculation

We have not yet computed any nonzero homotopy groups $\pi_n(X)$ with $n \geq 2$. In Chapter 1 the two main tools we used for computing fundamental groups were van Kampen's theorem and covering spaces. In the present section we will study the higher-dimensional analogs of these: the excision theorem for homotopy groups, and fiber bundles. Both of these are quite a bit weaker than their fundamental group analogs, in that they do not directly compute homotopy groups but only give relations between the homotopy groups of different spaces. Their applicability is thus more

— Algebraic Topology by Allen Hatcher

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Seifert-van Kampen

Covering spaces

Blakers-Massey

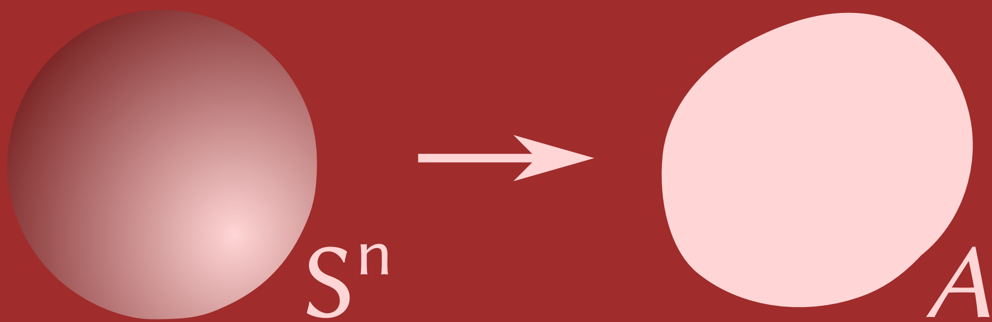
— Algebraic Topology by Allen Hatcher

favonia #1

472 commits / 69,881 ++ / 55,833 --

Homotopy Groups

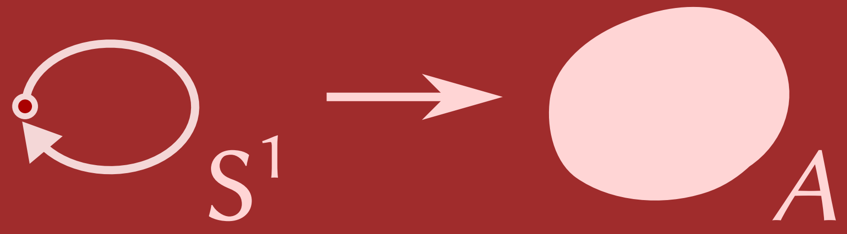
{ mappings from the *n*-sphere }



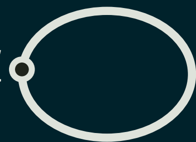
"higher" if $n > 1$

First Homotopy Group

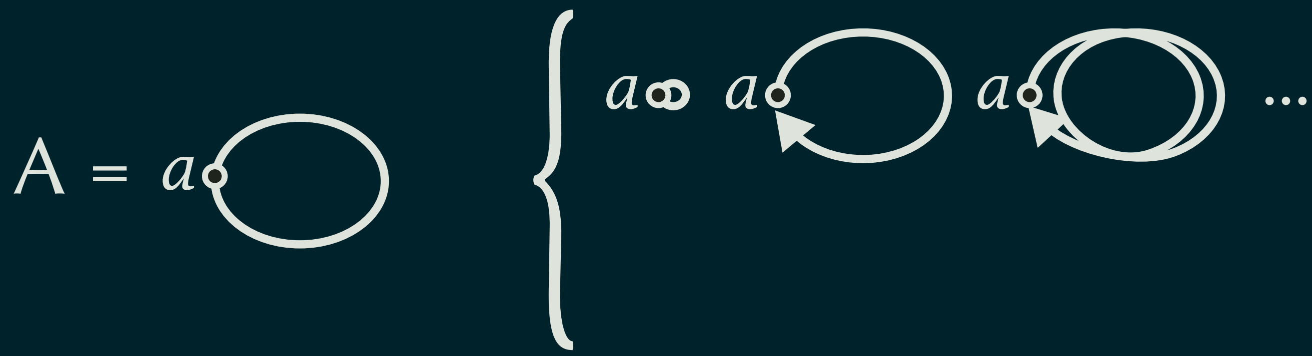
{ mappings from the circle }

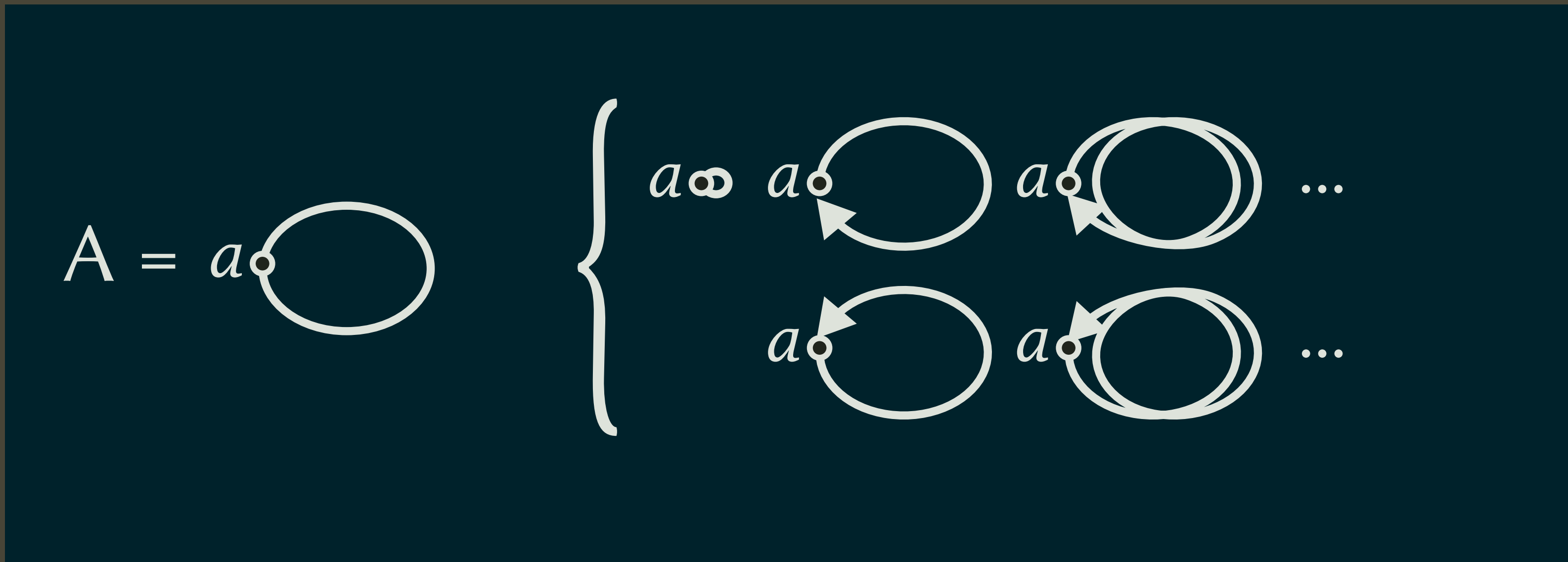


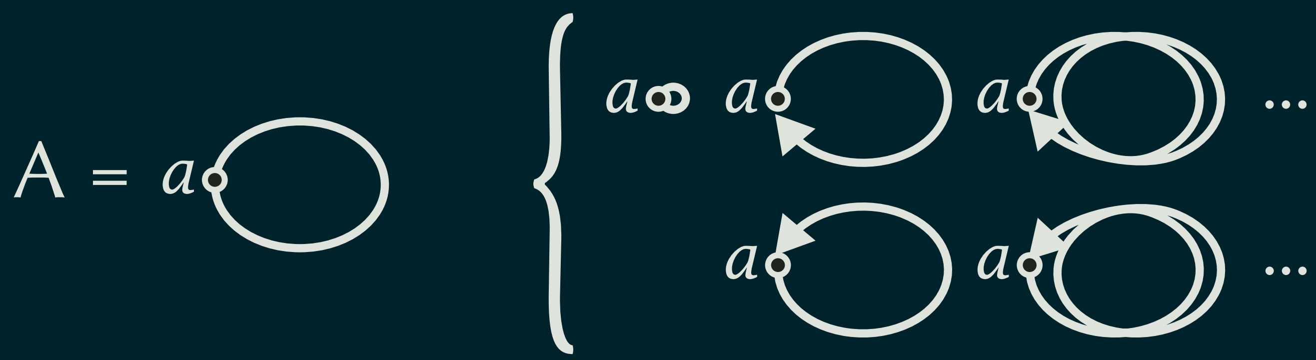
directed loops at some point

$A = a$ 


A =
a
a


A = a
a
a
a
...


A = a
a a a ...
a a ...



(much more)

Sets with Actions

Instead of computing these groups directly, consider sets with an *action* by the groups

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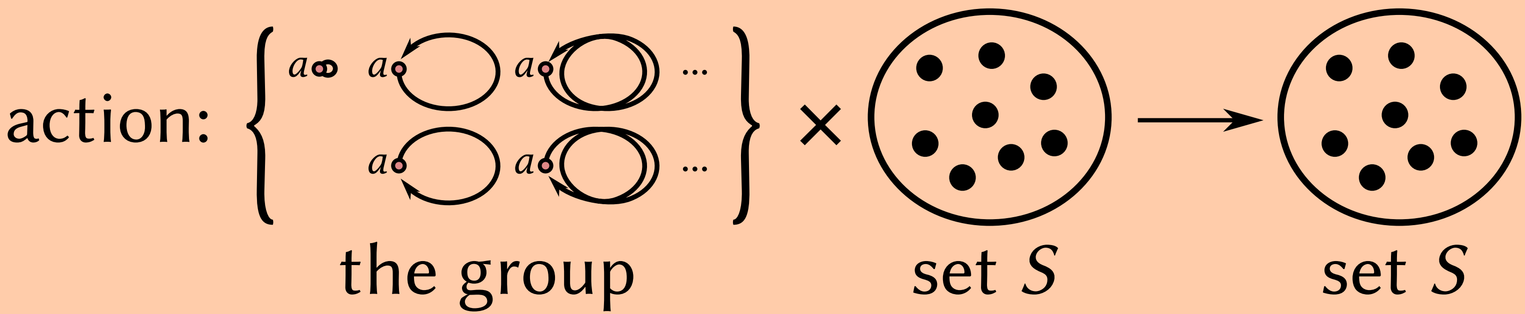
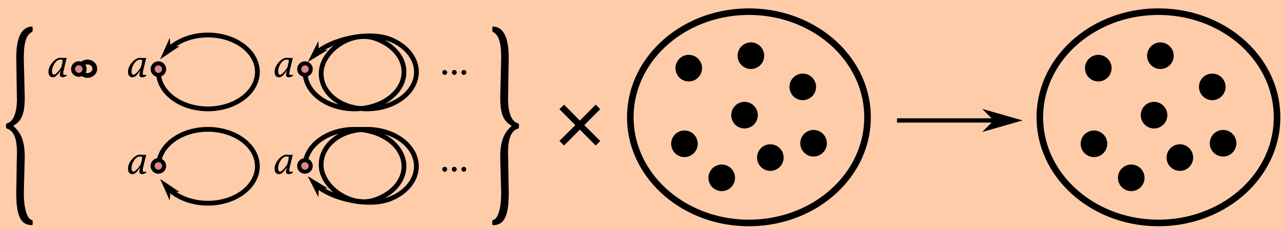


Sets with Actions

Instead of computing these groups directly, consider sets with an *action* by the groups

action: 

the group — set S — set S

Subject: *sets* with *some action by the first homotopy group*

Sets with some action by the first homotopy group

Sets with some action by the first homotopy group

≅

Covering spaces

Covering Spaces

Classical definition

A covering space of A is a space C together with a continuous surjective map $p : C \longrightarrow A$, such that for every $a \in A$, there exists an open neighborhood U of a, such that $p^{-1}(U)$ is a union of disjoint open sets in A, each of which is mapped homeomorphically onto U by p.

Type-theoretic definition

$$F : A \longrightarrow Set$$

Theorem*

Sets with some action by the first homotopy group of A

≀

Covering spaces

$$F : A \longrightarrow Set$$

More results in "Higher Groups in Homotopy Type Theory"

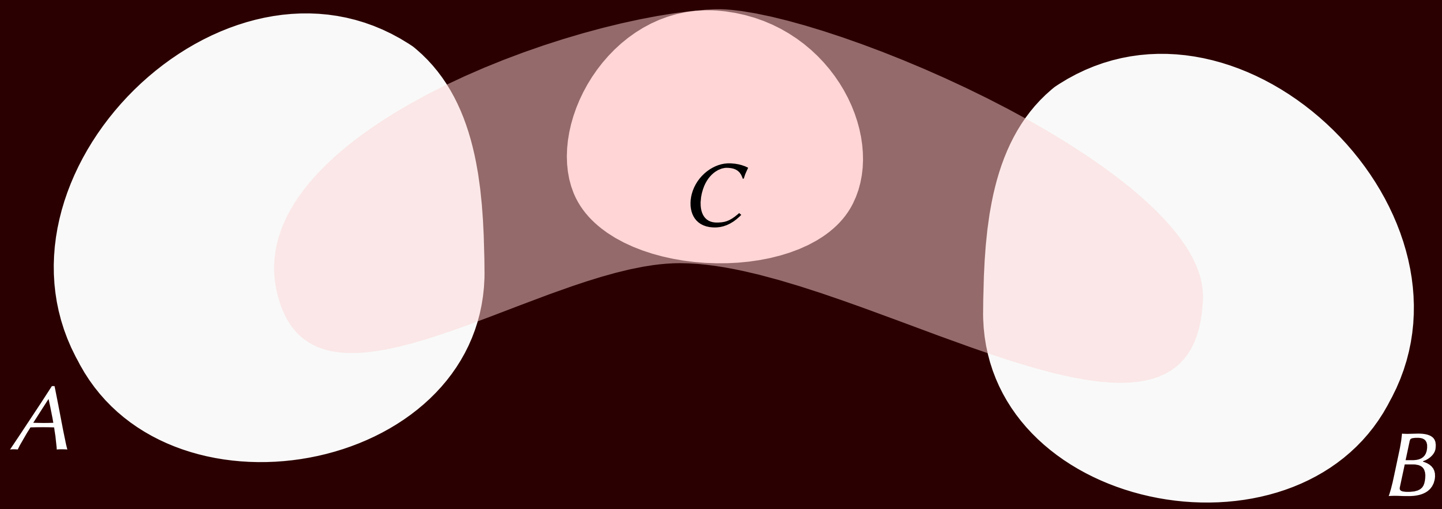
**A is pointed and connected*

Pushouts

Disjoint sums + gluing

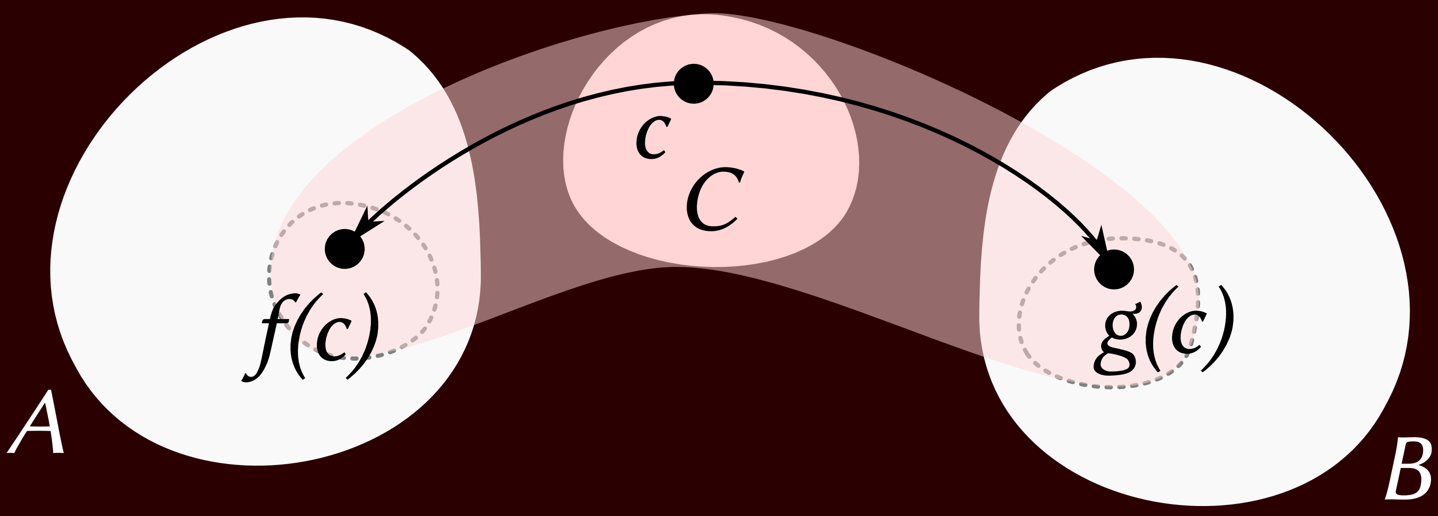
Pushouts

Disjoint sums + gluing



Pushouts

Disjoint sums + gluing



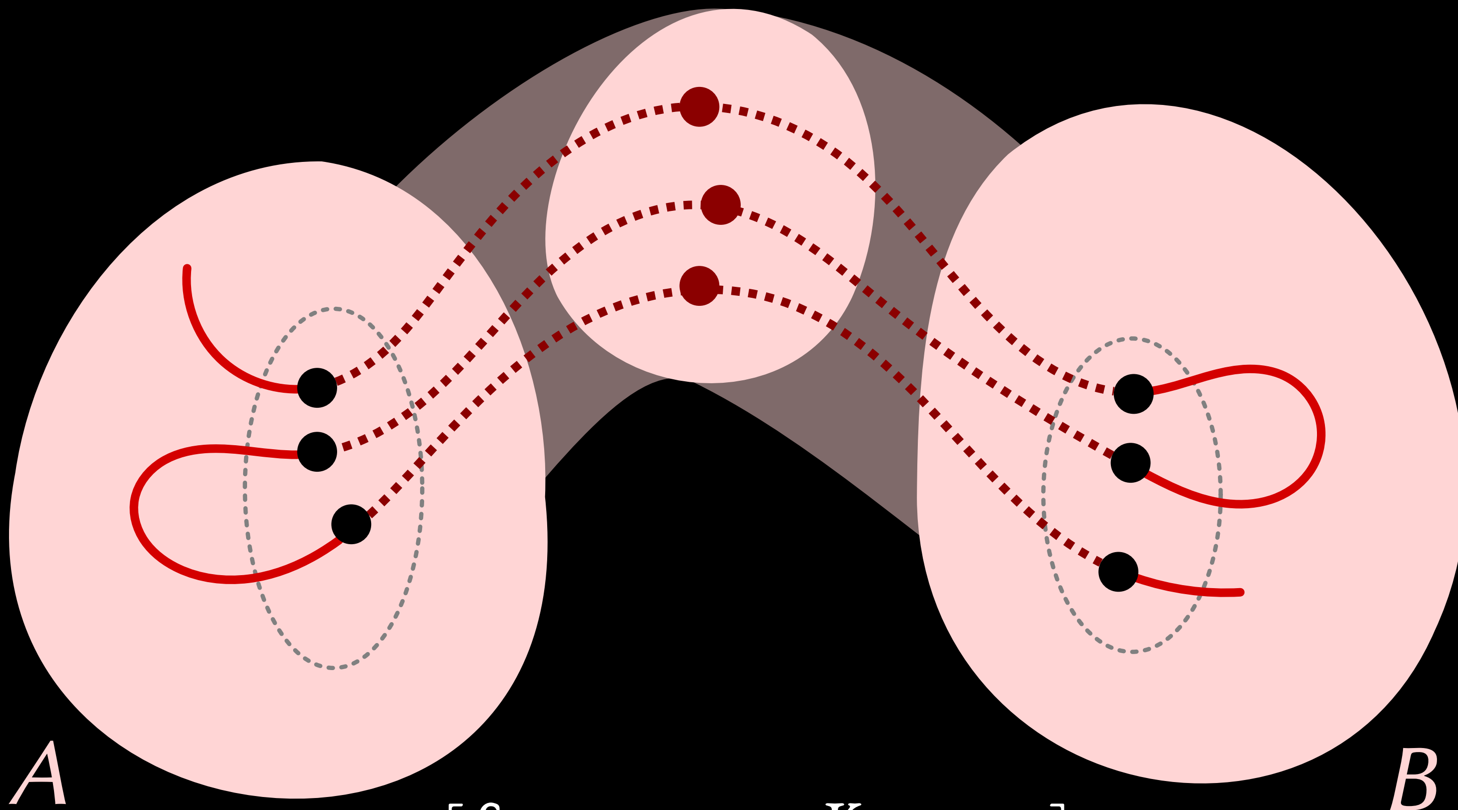
First Homotopy Groups

all paths from a point to itself

[GENERALIZATION]

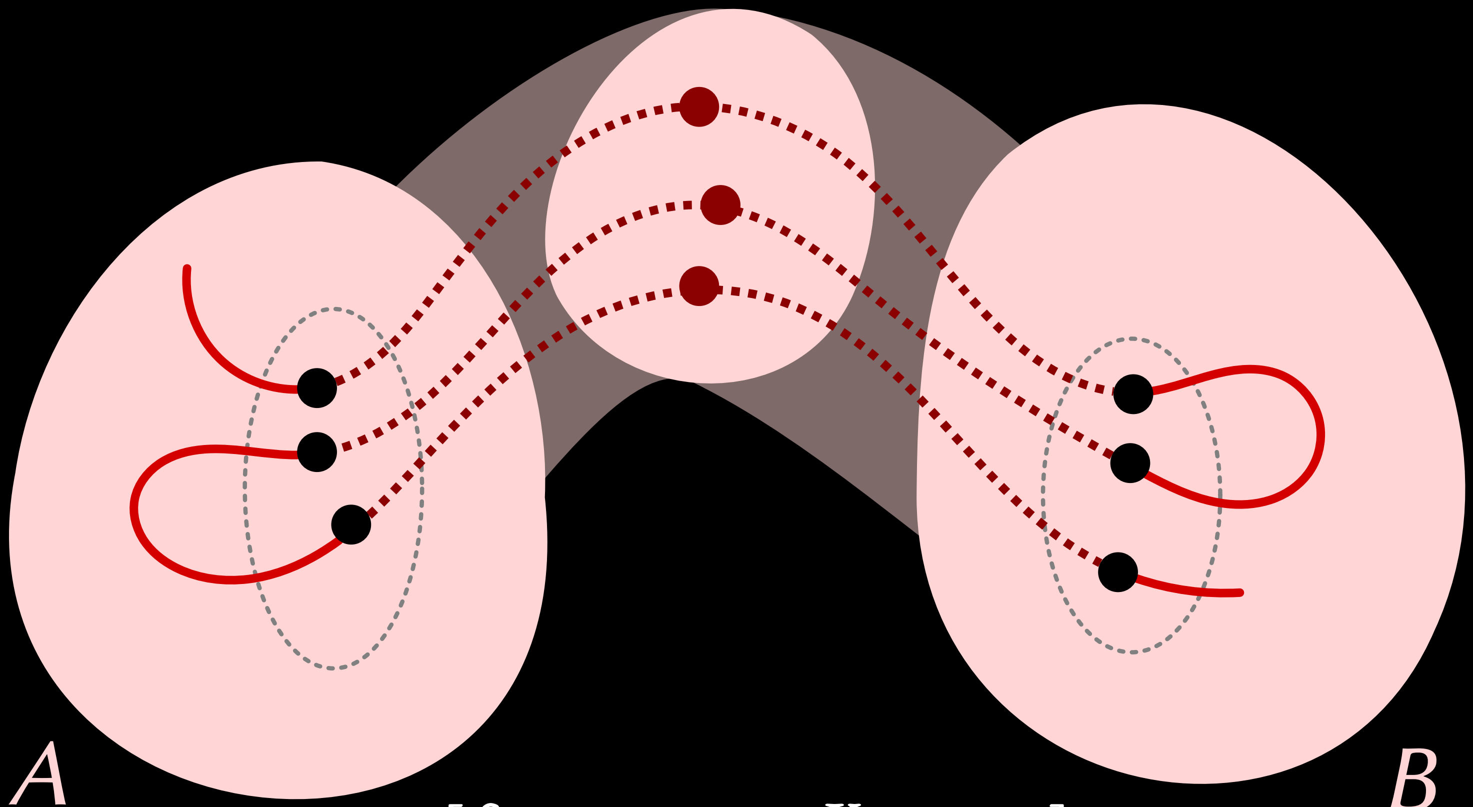
Fundamental Groupoids

all paths between any two points



[Seifert–van Kampen]

All paths are sequences of alternating paths in A and B



[Seifert–van Kampen]

Paths of the pushout can be calculated from paths of A and B and points of C

Homotopy Groups of Spheres

	1	2	3	4	5	6	7	8	9	10
S^1	Z	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^4	0	0	0	Z	Z_2	Z_2	$Z \times Z_{12}$	$Z_2{}^2$	$Z_2{}^2$	$Z_{24} \times Z_3$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0

Homotopy Groups of Spheres

	1	2	3	4	5	6	7	8	9	10
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S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^4	0	0	0	Z	Z_2	Z_2	$Z \times Z_{12}$	Z_2^2	Z_2^2	$Z_{24} \times Z_3$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2
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S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
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S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^4	0	0	0	Z	Z_2	Z_2	$Z \times Z_{12}$	$Z_2^{\ 2}$	$Z_2^{\ 2}$	$Z_{24} \times Z_3$
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S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0

corollary of [Blakers-Massey]

Blakers-Massey in homotopy type theory

[Finster, Licata, Lumsdaine] (2012-13)

Blakers-Massey in homotopy type theory

[Finster, Licata, Lumsdaine] (2012-13)

Full mechanization of Blakers-Massey in Agda

(2013) [Licata?] [Favonia]

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Un-mechanization into classical theory

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Generalization

[Anel, Biedermann, Finster, Joyal] (2016 or earlier)

Generalization available on arXiv

1703.09050 (2017)

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Generalization available on arXiv

1703.09050 (2017)

Full mechanization of Blakers-Massey in Agda

(2013) [Licata?] [Favonia]

Mechanization published

(2016) [FFLL]

Mechanization of the generalization in Agda?

(2017-?)

Charles Rezk +6

Oct 29, 2014

So, I believe I understand the proof of Blakers-Massey as coded by +favonia mlatus . At least, I understand enough of it that I can fill in the rest.

It is a most excellent proof!

Here are a few things I've learned.

1. The theorem involves the pushout P of a diagram X <-- f -- Q

Homology Groups

{ holes in a space }

Cohomology Groups

{ mappings from holes in a space }

Homology Groups

{ holes in a space }

Cohomology Groups

{ mappings from holes in a space }

Easier than homotopy groups
for many spaces of interest

Homology Groups of Spheres

	1	2	3	4	5	6
S^1	Z	0	0	0	0	0
S^2	0	Z	0	0	0	0
S^3	0	0	Z	0	0	0
S^4	0	0	0	Z	0	0
S^5	0	0	0	0	Z	0
S^6	0	0	0	0	0	Z

[STATEMENT]

Higher-dimensional types

provide novel abstraction that facilitates
the mechanization of homotopy theory

↑

[EXPERIMENTS]

Covering spaces
Seifert-van Kampen theorem
Blakers-Massey theorem
Cohomology groups

Post-Thesis

Post-Thesis

Involved in the development of *cubical type theory*

Post-Thesis

Involved in the development of *cubical type theory*

Ask Bob and his students

Thanks to Bob, the PoP group,
the HoTT community, CMU staff
(esp. Deborah and Catherine),
my spouse, and many, many people

...also Mark Rothko
for artistic inspiration

Have fun!