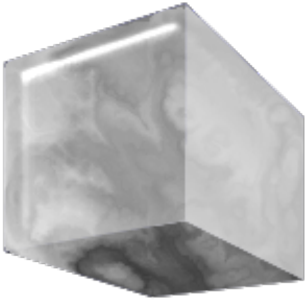
2018.02.07 Cornell

Cubical Computational Type Theory & RedPRL

>> redprl.org >>



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Evan Cavallo
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Dan Licata
Jon Sterling
Todd Wilson

Vladimir Voevodsky
1966-2017

Martin Hofmann
1965-2018

Cubical & Computational

Features

1. computational: canonicity by definition
2. key features from homotopy type theory (HoTT)
3. exact equality types

Cubical & Computational

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1. computational: canonicity by definition
2. key features from homotopy type theory (HoTT)
3. exact equality types

Advantages for computer science:

1. Equational reasoning closer to standard mathematics
2. Equivalent (good) types share the same properties
3. Quotients and other types built with relations
4. Openness to new constructs

Computational Types

$\boxed{\text{programs/ realizers}}$

computation

Computational Types

programs/ realizers	<------	computational type theory
computation		theory of computation

Computational Types

programs/ realizers <------ **computational type theory**

computation — theory of computation

meaning explanation <----- Martin-Löf type theory

pre-mathematical in M-L's work

A Minimum Example

```
M := a | bool | true | false | if(M,M,M)
```

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bool val    true val    if(true,M,_) ↦ M
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The Language

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The Language

What are the types in canonical forms? **{bool}**

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What are the canonical forms of the types?

bool: {true, false}

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What are the types in canonical forms? **{bool}**

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bool: {true, false}

How they are equal? **syntactic equality**

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One Theory

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```
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```

types: {bool} with syntactic equality

bool: {true, false} with syntactic equality

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$A \doteq B$ type

$A \Downarrow A'$ $B \Downarrow B'$ and $A' \approx B'$

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$\text{bool} \doteq \text{bool}$ type

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$$M \doteq N \in A$$

$A \doteq A$ type, $M \Downarrow M'$, $N \Downarrow N'$, $A \Downarrow A'$ and $M' \approx_{A'} N'$

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$$\text{false} \doteq \text{false} \in \text{bool}$$

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$$\text{false} \doteq \text{false} \in \text{bool}$$

$$\underset{\Downarrow \text{true}}{\text{if(true,true,bool)}} \doteq \text{true} \in \underset{\Downarrow \text{bool}}{\text{if(true,bool,bool)}}$$

A Minimum Example

```
M := a | bool | true | false | if(M,M,M)
```

`types: {bool}` with syntactic equality

`bool: {true, false}` with syntactic equality

$$a{:}A \gg M \doteq N \in B$$

$P \doteq Q \in A$ implies $M[P/a] \doteq N[Q/a] \in B$

A Minimum Example

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$b{:}bool \gg b \doteq if(b,true,false) \in bool$?

Variables

In Nuprl and friends

variables range over closed terms

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In Coq, Agda, and friends

variables (generally)
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---

closed reduction ⇔ vars over closed terms
open reduction ⇔ vars indeterminate

A Functional Example

```
M := a | M1⟶M2 | \a.M | M1 M2 | ...

(M1⟶M2) val  \a.M val  (\a.M1)M2 ↦ M1[M2/a]
```

Another Language

A Functional Example

```
M := a | M1→M2 | \a.M | M1 M2 | ...

(M1→M2) val   \a.M val   (\a.M1)M2 ↦ M1[M2/a]
```

Another Language

What are the types in canonical forms?

```
the least fixed point of
S.({M→N | M⇓, N⇓ in S} union ...)
```

What are the canonical forms of the types?

```
A→B: {\a.M}
```

How they are equal?

```
A1→B1 ≈ A2→B2 if A1 ≐ A2 and B1 ≐ B2
```

\a.M1 $\approx_{A\rightarrow B}$ \a.M2 if a:A >> M1 ≐ M2 ∈ B

Openness

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(rules in proof assistants)
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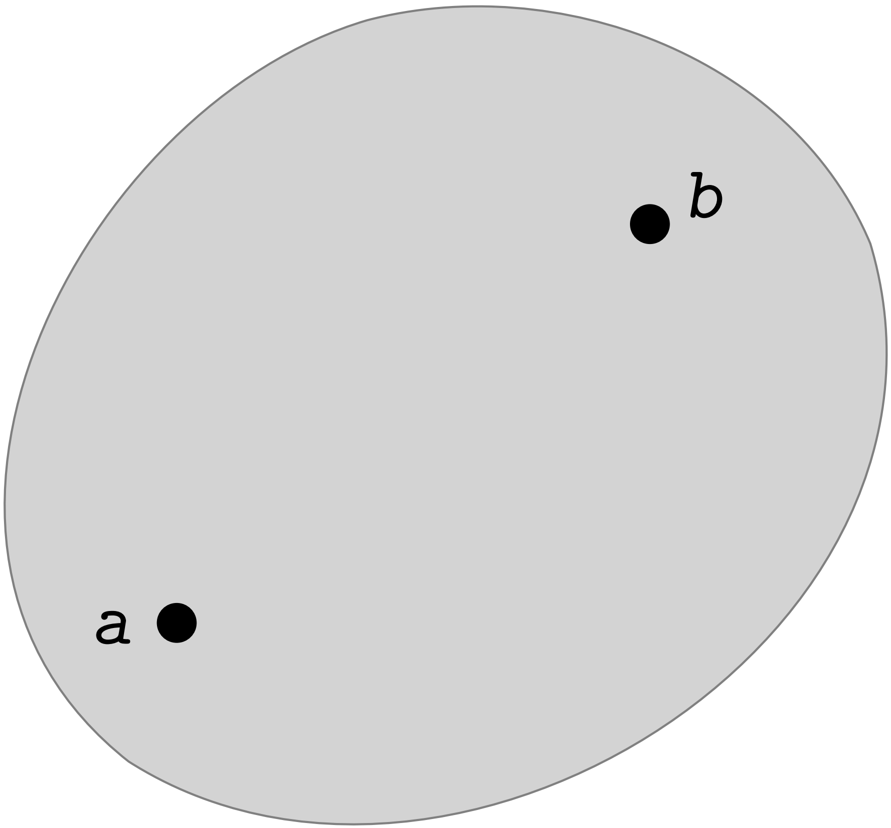
Canonicity always holds

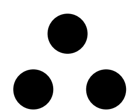
Homotopy Type Theory

[Awodey and Warren] [Voevodsky *et al*] [van den Berg and Garner]

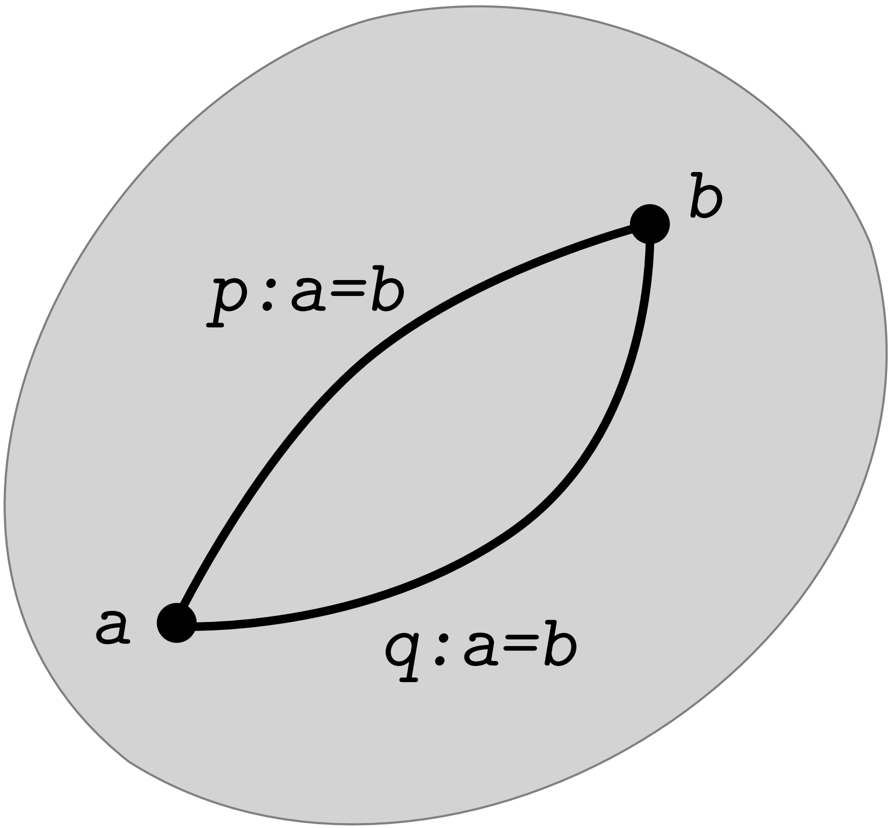
A	Type	Space
$a : A$	Element	Point
$f : A \to B$	Function	Continuous Mapping
$C : A \to Type$	Dependent Type	Fibration
$a =_A b$	Identification	Path

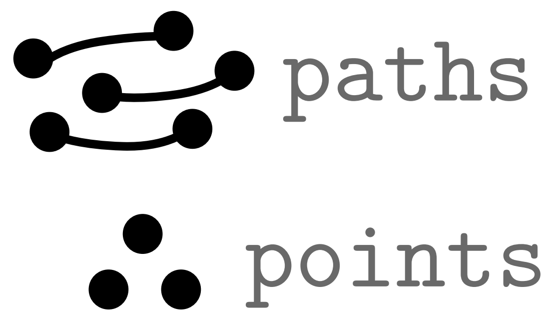
Homotopy Type Theory



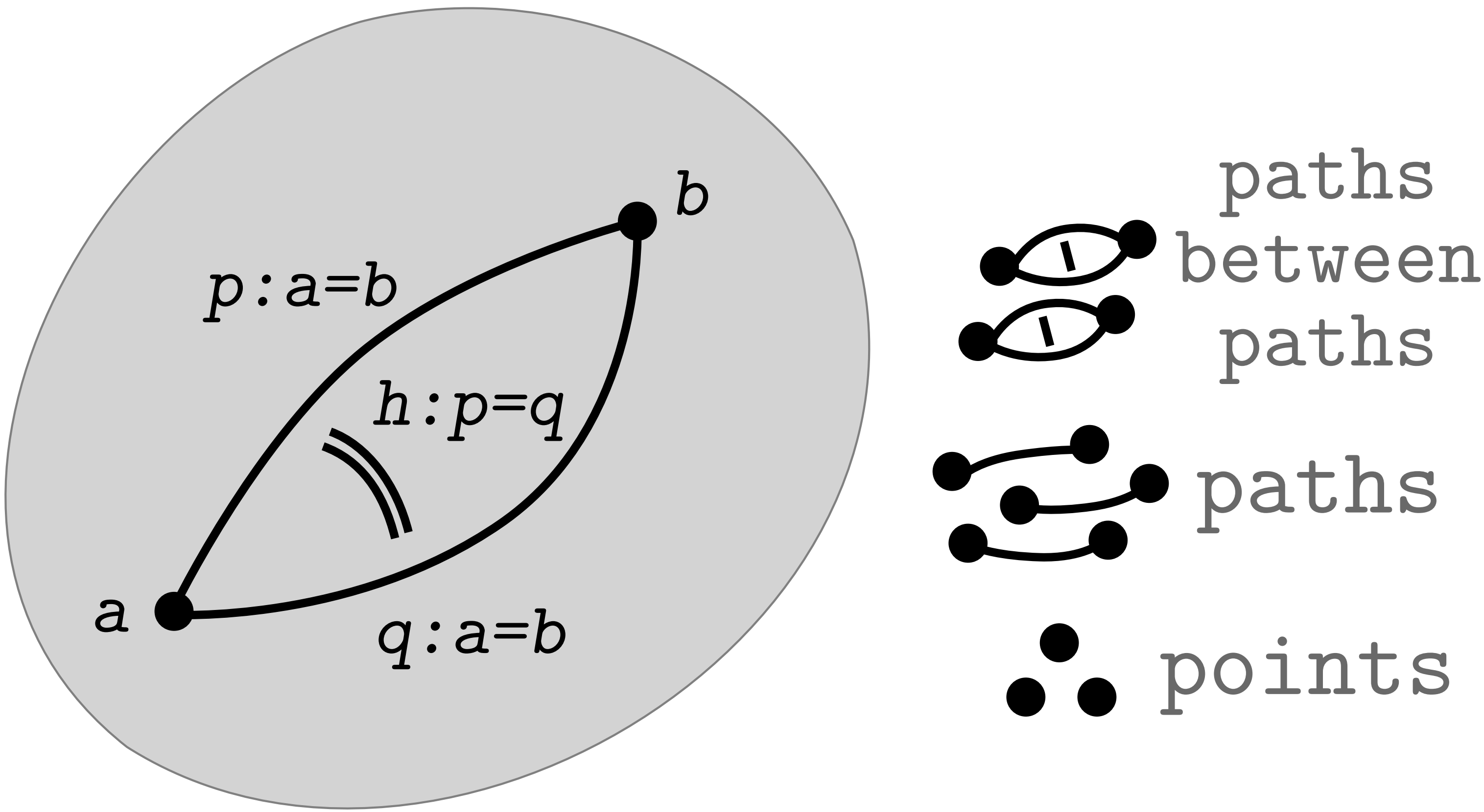


Homotopy Type Theory





Homotopy Type Theory



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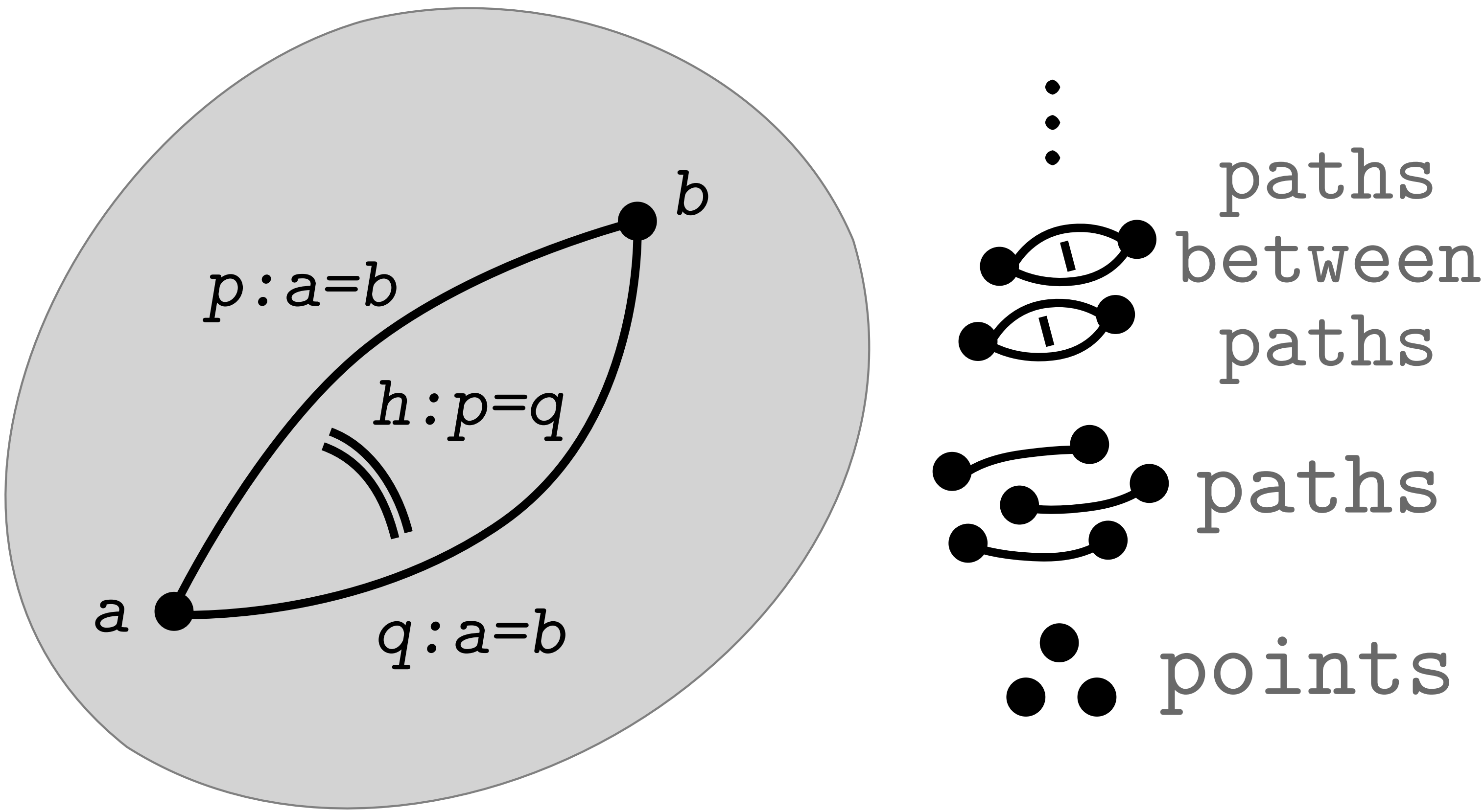

Homotopy Type Theory

Tons of results in homotopy theory are mechanized through this.

In some case new proofs were discovered and inspired new results.
[Anel, Biedermann, Finster, Joyal]

Also lots of works in category theory and other fields.

Key Features of HoTT

0. Based on intensional type theory
1. Identifications as paths
2. Univalence: if e is an equivalence between A and B, then $ua(e):A=B$
3. Higher inductive types: generalized inductive types with (higher) path generators

Key Features of HoTT

0. Based on intensional type theory

1. Identifications as paths

2. Univalence: if e is an equivalence between A and B, then $ua(e) : A = B$

3. Higher inductive types: generalized inductive types with (higher) path generators

Problems: 2&3 give new identifications

The Poor J Eliminator

$J[a.C]$(refl-case, path)

eliminator for identifications
can only handle reflexivity

$coe(p : A = B, a : A) : B$

$coe(ua(e), a)$ is stuck

The Poor J Eliminator

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$coe(p{:}A{=}B, a{:}A){:}B$

$coe(ua(e), a)$ is stuck

Solution

motive C handles paths itself

The Happy J Eliminator

each motive handles paths itself

each type has cubical Kan structure

[Bezem, Coquand, Huber] [Cohen, Coquand, Huber, Mörtberg]

This work:
extend Nuprl by cubical Kan structures

[Angiuli, Harper, Wilson] [Angiuli, Harper] [Angiuli, Favonia, Harper] [Cavallo, Harper]

Cubical Programming

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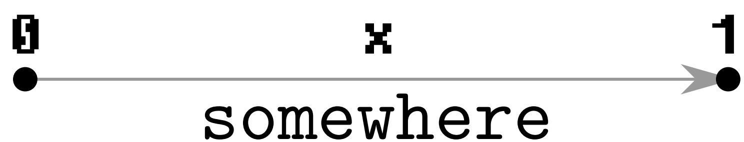
```
dim expr r := 0 | 1 | x
```

Cubical Programming

`dim expr r := 0 | 1 | x`

dimension variables (x)
should not be inductively analyzable



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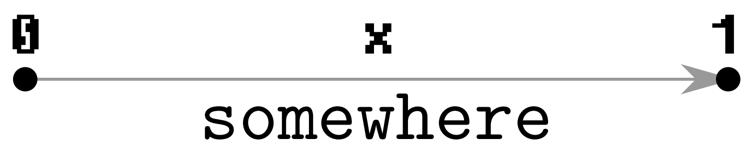
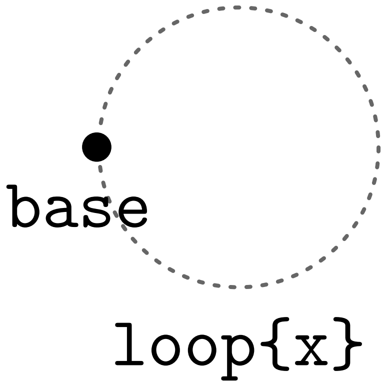


* new reduction *

closed in expression variables
open in dimension variables

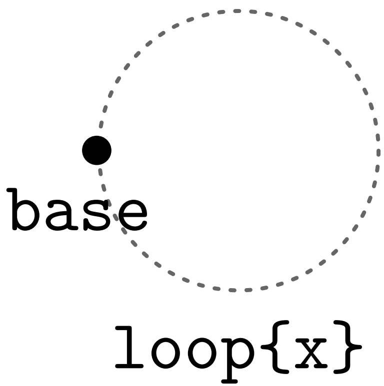
Circle



Circle

dim expr

```
M := S1 | base | loop{r}
   | S1elim(a.M, M, M, x.M) | ...
```



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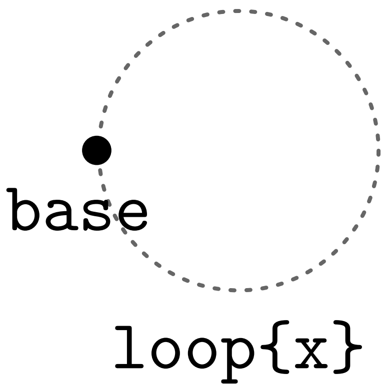
dim
expr



S1 val

Circle

```
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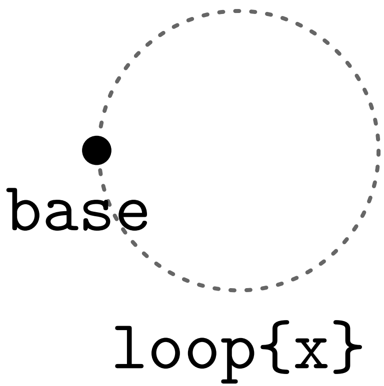
dim expr

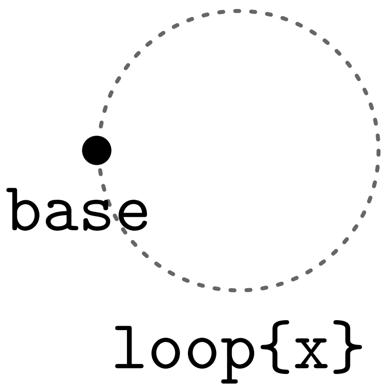


`base val`

`S1 val`

Circle

```
M := S1 | base | loop{r}   dim expr
   | S1elim(a.M, M, M, x.M) | ...
```



S1 val

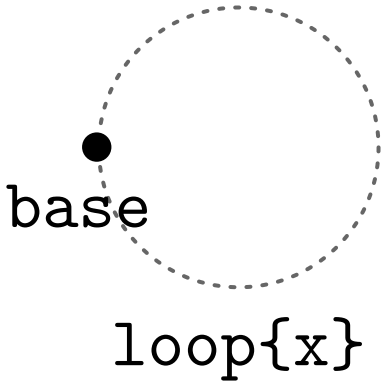
base val

loop{x} val

loop{0} ↦ base

loop{1} ↦ base

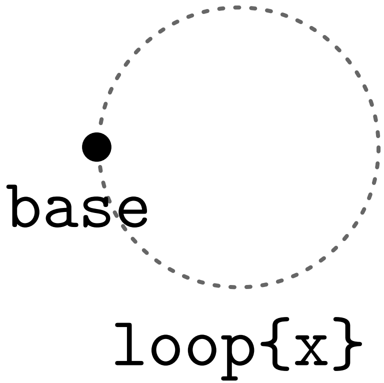
Circle




```
S1 val
```

```
M ↦ M'
────────────────────────────
S1elim(a.A, M, B, x.L)
  ↦ S1elim(a.A, M', B, x.L)
```

Circle



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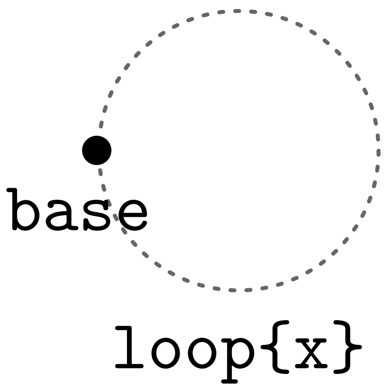


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```

```
S1elim(a.A, base, B, x._)
  ↦ B
```

```
S1elim(a.A, loop{x}, _, y.L)
  ↦ L<x/y>
```

Kan: Coercion

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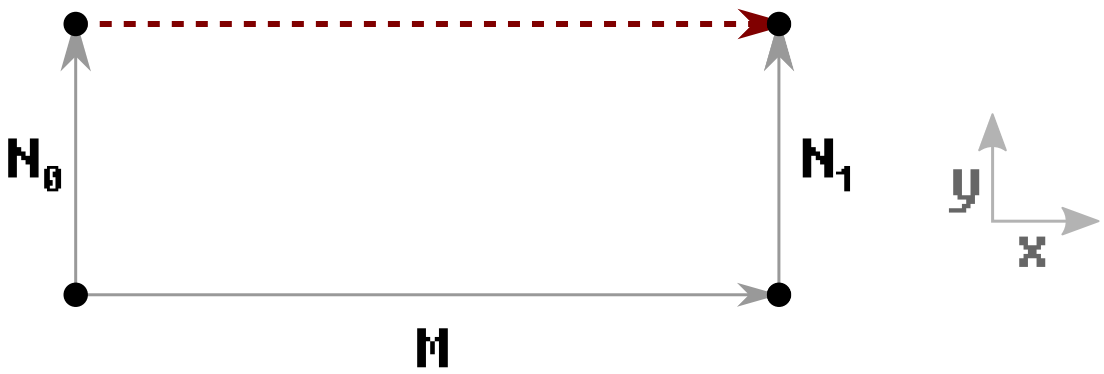
A<0/x> → A<1/x>

M → coe{0⟶1}(x.A,M)

coe{r⟶r'}(x.A, M) ∈ A<r'/x>

M ∈ A<r/x>

Kan: Homogeneous Comp.



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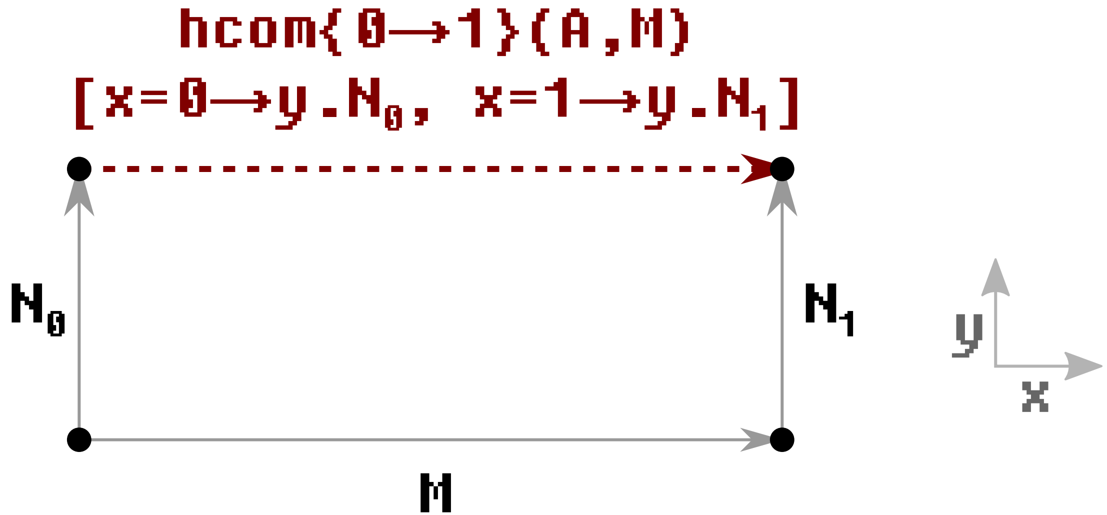

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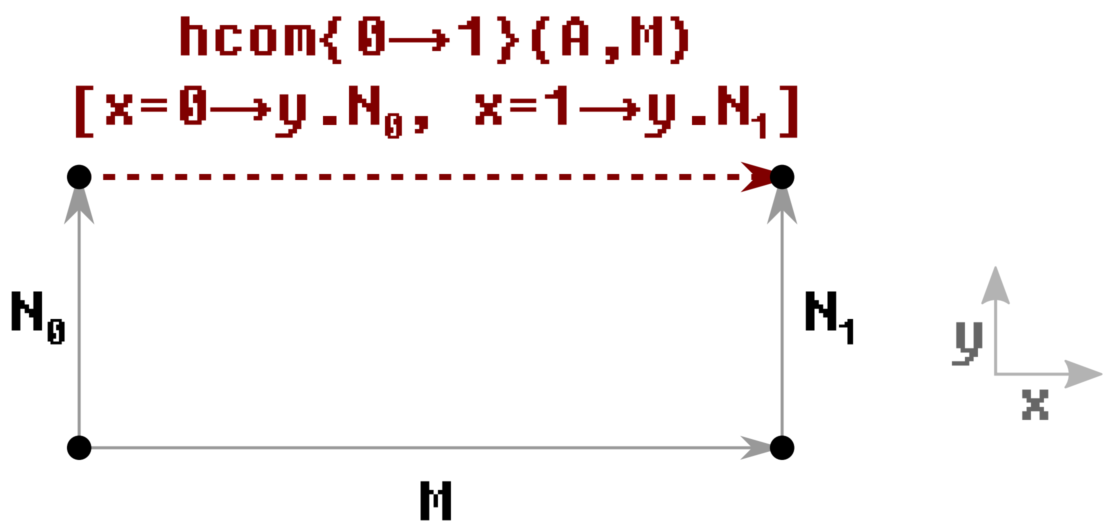


```
hcom{r⟶r'}(A, M)
  [..., rᵢ=r'ᵢ⟶y.Nᵢ, ...]
```

Kan Circle

```
coe{r⟶r'}(_.S1, M) ↦ M
```

Kan Circle

```
coe{r⟶r'}(_.S1, M) ↦ M
```

```
hcom{r⟶r'}(S1, M)[...] ↦ fcom{r⟶r'}(M)[...]
```

formal
composition

Kan Circle

```
coe{r⟶r'}(_.S1, M) ↦ M
```

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hcom{r⟶r'}(S1, M)[...] ↦ fcom{r⟶r'}(M)[...]
```

formal
composition

```
fcom{r⟶r}(M)[...] ↦ M
```

Kan Circle

```
coe{r→r'}(_.S1, M) ↦ M
```

```
hcom{r→r'}(S1, M)[...] ↦ fcom{r→r'}(M)[...]
```

formal composition

```
fcom{r→r}(M)[...] ↦ M
```

$$\frac{r \neq r' \quad r_i = r'_i \text{ (the first i)}}{\texttt{fcom\{r→r'\}(M)[..., } r_i = r'_i \texttt{→y.}N_i\texttt{, ...]} \mapsto N_i\langle r'/y\rangle}$$

Kan Circle

```
coe{r⟶r'}(_.S1, M) ↦ M
```

```
hcom{r⟶r'}(S1, M)[...] ↦ fcom{r⟶r'}(M)[...]
```

formal
composition

```
fcom{r⟶r}(M)[...] ↦ M
```

```
r≠r'   r_i=r'_i (the first i)
─────────────────────────────────────────────────
fcom{r⟶r'}(M)[..., r_i=r'_i⟶y.N_i, ...] ↦ N_i<r'/y>
```

```
r≠r'   r_i≠r'_i for all i
────────────────────────────
fcom{r⟶r'}(M)[...] val
```

Kan Circle

`S1elim` needs to handle `fcom`

Kan Circle

S1elim needs to handle **fcom**

```
r≠r'   r_i≠r'_i
──────────────────────────────────────
S1elim(a.A, fcom{r→r'}(M)[...], B, x.L)
  ↦ com{r→r'}(y.A[fcom{r→y}(...).../a],
          S1elim(M, B, x.L))[...]
```

S1elim(composition) ↦ composition(S1elim)

Cubical Stability

Dimension substs. do not commute with evaluation!

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$$
\begin{tikzcd}
\texttt{S1elim(a.A, loop\{x\}, B, y.L)} \arrow[r, mapsto, dashed] & \texttt{L<x/y>} \arrow[d, dashed, "\texttt{<0/x>}"] \\
& \texttt{L<0/y>}
\end{tikzcd}
$$

Cubical Stability

Dimension substs. do not commute with evaluation!

S1elim(a.A, loop{x}, B, y.L) |——→ L<x/y>

↓ <0/x> ↓ <0/x>

S1elim(a.A, base, B, y.L) |—> B <=??=> L<0/y>

Cubical Stability

Dimension substs. do not
commute with evaluation!

```
S1elim(a.A,
 loop{x}, B, y.L)   |-----------> L<x/y>
                                     |
     |                               |  <0/x>
     |  <0/x>                        v
     v

S1elim(a.A,
 base, B, y.L)  |--> B   <=??=>   L<0/y>
```

Restrict our theory to
only cubically stable terms

Cubical Type Theory

stability: consider every substitution

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stability: consider every substitution

$$A \doteq B \text{ type } [\Psi]$$

dim context

under any dim substitution ψ...
Aψ and Bψ stably* eval to A' and B' which stably* recognize the same stable* values and have stably* equal Kan structures

(see our arXiv and POPL papers)

Cubical Type Theory

stability: consider every substitution

$$A \doteq B \text{ type } [\Psi]$$

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under any dim substitution ψ...
Aψ and Bψ stably* eval to A' and B' which stably* recognize the same stable* values and have stably* equal Kan structures

$$M \doteq N \in A \ [\Psi]$$

A ≐ A type, A⇓A',
M and N stably* eval to M' and N',
A' stably* views N' and M' as the same value

(see our arXiv and POPL papers)

Our arXiv Papers

Part1: stability

Part2: dependent types

Part3: univalence and equality

Part4: cubical inductive types

RedPRL

a proof assistant based
on the new type theory

still nascent, changing everyday

http://redprl.org

Conclusion

We extended Nuprl semantics
by cubical Kan structures which
justify key features of HoTT

We also built RedPRL as a prototype