Cartesian Cubical Computational Type Theory

Favonia
Inst for Adv Study $\mapsto$ U of Minnesota

Joint work with Carlo Angiuli, Evan Cavallo,
Daniel R. Grayson, Robert Harper and Jonathan Sterling

(a shameless rip-off of Carlo's previous talks)

Bonn, Germany, 2018/6/5

Some History

Coquand's notes
BL 14

AHW 2016: CHTT Part I
Cartesian cubical + computational

AH 2017: CHTT Part II
Dependent types

AFH 2017: CHTT Part III
Univalent Kan universes

CH 2018: CHTT Part IV
Higher inductive types
Identification types

Some History

Coquand's notes
BL 14

AHW 2016: CHTT Part I
Cartesian cubical + computational

AH 2017: CHTT Part II
Dependent types

AFH 2017: CHTT Part III
Univalent Kan universes

CH 2018: CHTT Part IV
Higher inductive types
Identification types

ABCFHL 2017: CCTT
Cartesian cubical

Some History

Coquand's notes
BL 14

AHW 2016: CHTT Part I
Cartesian cubical + computational

AH 2017: CHTT Part II
Dependent types

AFH 2017: CHTT Part III
Univalent Kan universes

CH 2018: CHTT Part IV
Higher inductive types
Identification types

ABCFHL 2017: CCTT
Cartesian cubical

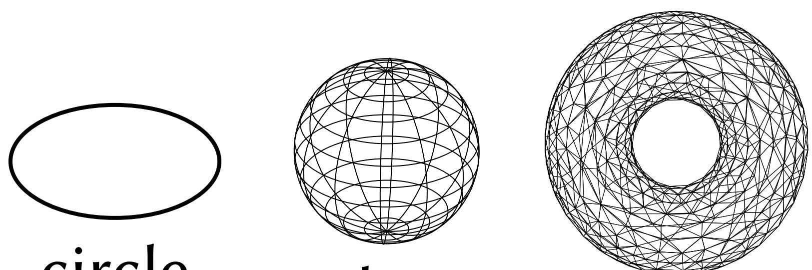
New Features of HoTT

Univalence

if e is an equivalence between
types A and B, then ua(E):A=B

Higher Inductive Types

inductive types with path generators



circle sphere torus

Equality and Paths

Definitional Equality

Silent in theory

$$2 + 3 \equiv 5$$
$$\mathrm{fst}\ \langle \mathrm{M}, \mathrm{N} \rangle \equiv M$$

Equality and Paths

Definitional Equality

Silent in theory

$2 + 3 \equiv 5$

$\mathrm{fst}\ \langle M, N \rangle \equiv M$

If $A \equiv B$ and $M : A$ then $M : B$

Equality and Paths

Definitional Equality

Silent in theory

$2 + 3 \equiv 5$

$\mathrm{fst}\ \langle M,N \rangle \equiv M$

If $A \equiv B$ and $M : A$ then $M : B$

Paths

Visible in theory

If $P : \mathrm{Path}(A, B)$ and $M : A$ then $\mathrm{transport}(M,P) : B$

Not Math Equality!

Definitional Equality Issue #1

Not very extensional

$x : \mathbb{N}, y : \mathbb{N} \vdash x + y \not\equiv y + x : \mathbb{N}$

(various reasonable trade-offs)

Not Math Equality!

Definitional Equality Issue #2

$$\pi_1(S1) \xrightarrow{\text{winding}} \mathbb{Z}$$

winding(loop) $\not\equiv$ any numeral

Not Math Equality!

Definitional Equality Issue #2

$$\pi_1(S1) \xrightarrow{\text{winding}} \mathbb{Z}$$

winding(loop) $\not\equiv$ any numeral

Canonicity

For any $M : \mathbb{N}$, there is a numeral N^* such that $\vdash M \equiv N^* : \mathbb{N}$

Restore Canonicity

Canonicity for $\mathbb{N}$ means
canonicity for *every* type

Restore Canonicity

Canonicity for $\mathbb{N}$ means canonicity for *every* type

$$M : \mathbb{N} \times A$$
$$\mathrm{fst}(M) \equiv ??? : \mathbb{N}$$

Restore Canonicity

Canonicity for $\mathbb{N}$ means canonicity for *every* type

$M : \mathbb{N} \times A$
$\mathsf{fst}(M) \equiv ??? : \mathbb{N}$

By canonicity of pair types, $M \equiv \langle P,Q \rangle$ and
$\mathsf{fst}(M) \equiv \mathsf{fst}\ \langle P,Q \rangle \equiv P \equiv$ some numeral
(by i.h.) (rule) (by i.h.)

Restore Canonicity

But canonicity fails for paths!

$$\mathrm{refl}(M) : \mathrm{Path}(M,M)$$

$$\text{Path-elim}[p.C](M, x.N) : C[M/p]$$

$$\text{Path-elim}(\mathrm{refl}(M), x.N) \equiv N[M/x] : C[\mathrm{refl}(M)/p]$$

Restore Canonicity

But canonicity fails for paths!

refl(M) : Path(M,M)

Path-elim[p.C](M, x.N) : C[M/p]

Path-elim(refl(M), x.N) ≡ N[M/x] : C[refl(M)/p]

Path-elim(ua(E), x.N) ≡ ???

Path-elim(loop, x.N) ≡ ???

Restore Canonicity

Can we have canonicity + univalence?

Yes with De Morgan cubes (CCHM)
Yes with Cartesian cubes (Part III by AFH)

And higher inductive types?

Important examples with De Morgan cubes (CHM)
Yes with cartesian cubes (Part IV by CH)

Cubes

Idea: each type manages its own paths

Cubes

Idea: each type manages its own paths

$$\text{loop} : \text{base} = \text{base}$$

Cubes

Idea: each type manages its own paths

loop : base = base

loop_x: a genuine constructor of *the circle*

$$x{:}\mathbb{I} \vdash \text{loop}_x : \text{S1}$$

$$\text{loop}_0 \equiv \text{base} : \text{S1} \qquad \text{loop}_1 \equiv \text{base} : \text{S1}$$

Cartesian Cubes

Introducing $\mathbb{I}$ the formal interval

Cartesian Cubes

Introducing $\mathbb{I}$ the formal interval

$$\Gamma \vdash 0 : \mathbb{I} \qquad \Gamma \vdash 1 : \mathbb{I}$$

$$\Gamma, x : \mathbb{I}, \Gamma' \vdash x : \mathbb{I}$$

Cartesian Cubes

Introducing $\mathbb{I}$ the formal interval

$$\Gamma \vdash 0 : \mathbb{I} \qquad \Gamma \vdash 1 : \mathbb{I}$$

$$\Gamma, x : \mathbb{I}, \Gamma' \vdash x : \mathbb{I}$$

$$x_1 : \mathbb{I}, x_2 : \mathbb{I}, \ldots, x_n : \mathbb{I} \vdash M : A$$

⇔ *M* is an n-cube in A

Cartesian Cubes

Introducing $\mathbb{I}$ the formal interval

$$\Gamma \vdash 0 : \mathbb{I} \qquad \Gamma \vdash 1 : \mathbb{I}$$

$$\Gamma, x : \mathbb{I}, \Gamma' \vdash x : \mathbb{I}$$

Cartesian: works as normal contexts

$$M\langle 0/x \rangle \qquad M\langle 1/x \rangle \qquad M\langle y/x \rangle$$

Ordinary Types

Ordinary typing rules hold uniformly

$$\frac{\Gamma,\ a{:}A \vdash M : B}{\Gamma \vdash \lambda a.M : (a{:}A) \longrightarrow B}$$

with any number of $\mathbb{I}$ in the Γ

Ordinary Types

Ordinary typing rules hold uniformly

$$\frac{\Gamma,\ a{:}A \vdash M : B}{\Gamma \vdash \lambda a.M : (a{:}A) \longrightarrow B}$$

with any number of $\mathbb{I}$ in the Γ

$$F(M_x\langle 0/x\rangle) \xrightarrow{\ \mathsf{ap}_F(M)\quad F(M_x)\ } F(M_x\langle 1/x\rangle)$$

New Path Types

Dimension abstraction

$$\frac{x{:}\mathbb{I} \vdash M : A}{\langle x\rangle M : \mathrm{Path}_{x.A}(M\langle 0/x\rangle, M\langle 1/x\rangle)}$$

$$\frac{P : \mathrm{Path}_{x.A}(N_0, N_1)}{P@r : A\langle r/x\rangle}$$

$$\frac{x{:}\mathbb{I} \vdash M : A}{(\langle x\rangle M)@r \equiv M\langle r/x\rangle : A\langle r/x\rangle}$$

$$\frac{P : \mathrm{Path}_{x.A}(N_0, N_1)}{P@0 \equiv N_0 : A\langle 0/x\rangle}$$

$$\frac{P : \mathrm{Path}_{x.A}(N_0, N_1)}{P@1 \equiv N_1 : A\langle 1/x\rangle}$$

Kan 1/2: Coercion

$$\frac{M : A\langle r/x\rangle}{\mathsf{coe}_{x.A}[r \rightsquigarrow r'](M) : A\langle r'/x\rangle}$$

Kan 1/2: Coercion

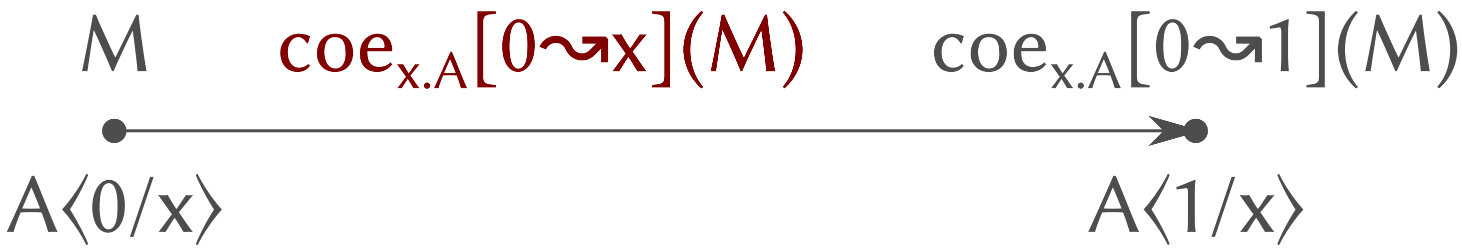
$$\frac{M : A\langle r/x\rangle}{\mathsf{coe}_{x.A}[r \rightsquigarrow r'](M) : A\langle r'/x\rangle}$$

Kan 1/2: Coercion

$$\frac{M : A\langle r/x\rangle}{\mathsf{coe}_{x.A}[r \rightsquigarrow r'](M) : A\langle r'/x\rangle}$$

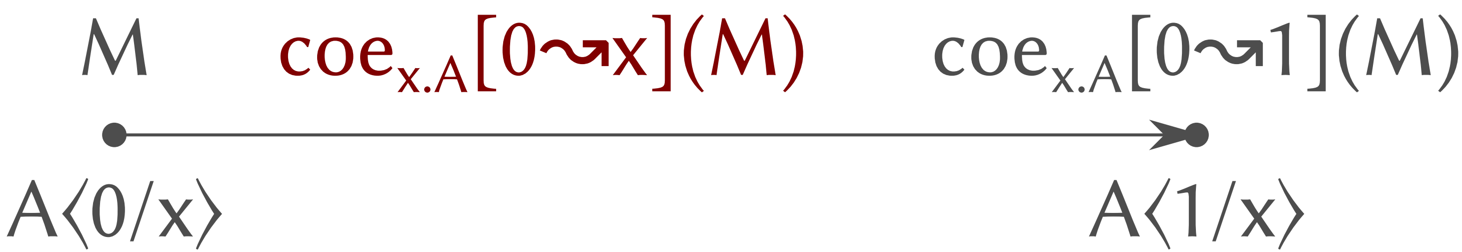
Kan 1/2: Coercion

$$\frac{M : A\langle r/x \rangle}{\mathsf{coe}_{x.A}[r \rightsquigarrow r'](M) : A\langle r'/x \rangle}$$



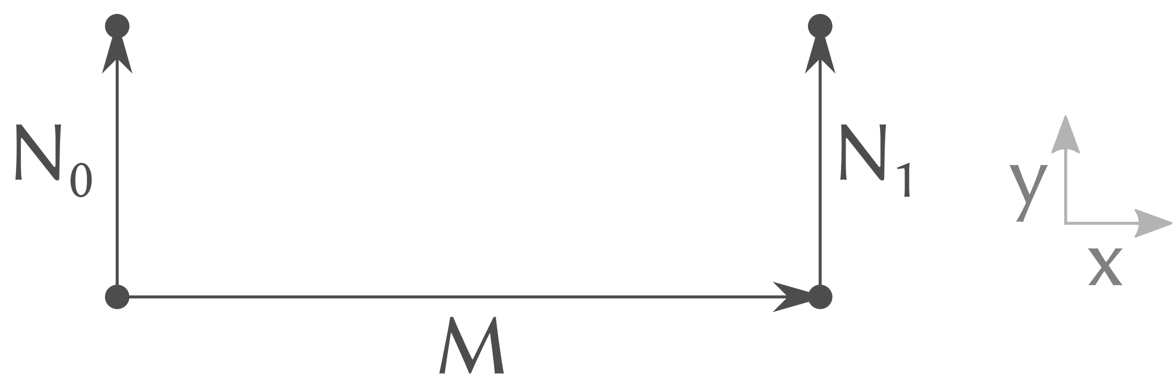
Kan 1/2: Coercion

$$\frac{M : A\langle r/x\rangle}{\mathsf{coe}_{x.A}[r \rightsquigarrow r'](M) : A\langle r'/x\rangle}$$



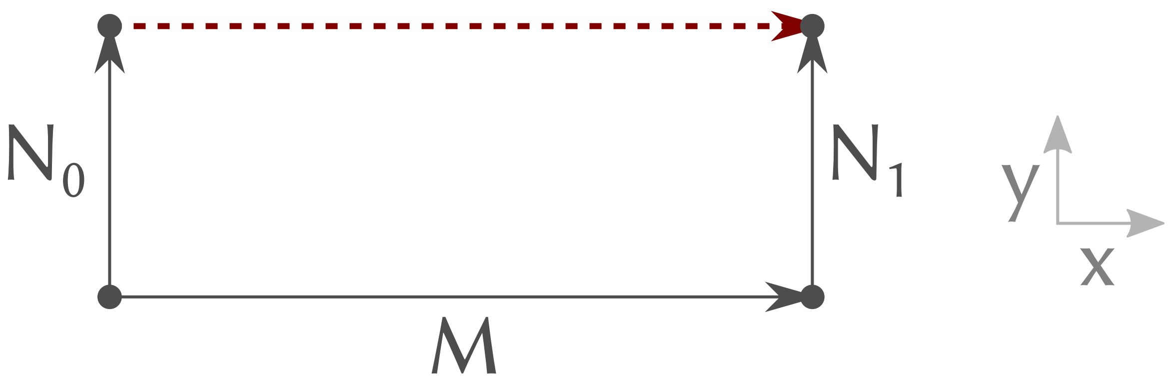
$$\mathsf{coe}_{x.A}[r \rightsquigarrow r](M) \equiv M : A\langle r/x\rangle$$

Kan 2/2: Homogeneous Comp



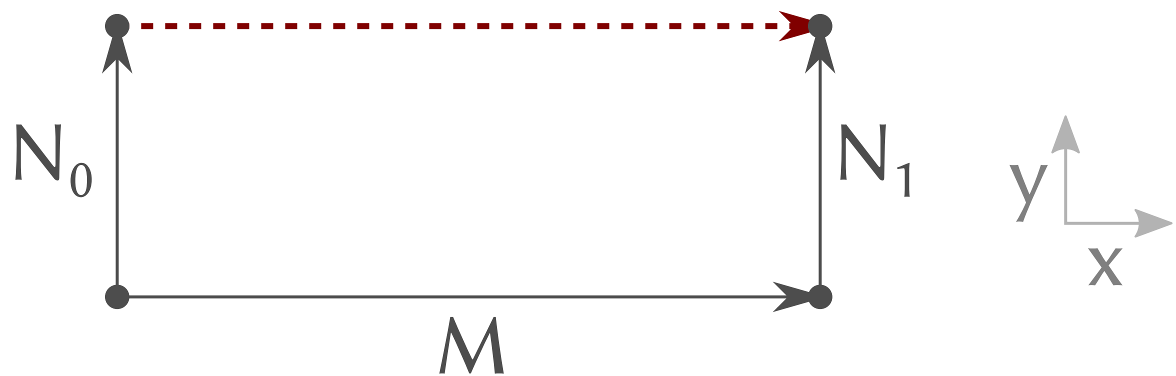
Kan 2/2: Homogeneous Comp

$\mathsf{hcom}_A[0 \rightsquigarrow 1](M;\ x{=}0 \hookrightarrow y.N_0,\ x{=}1 \hookrightarrow y.N_1)$



Kan 2/2: Homogeneous Comp

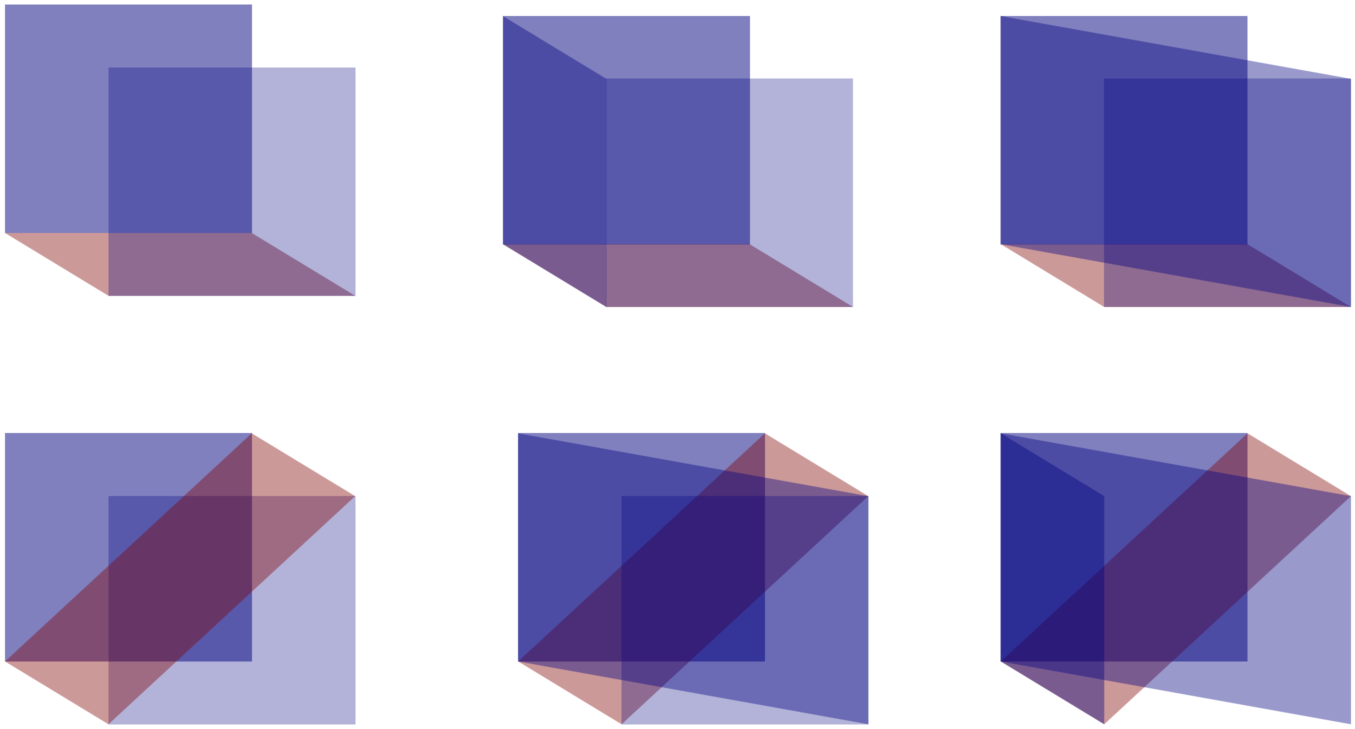
$\mathrm{hcom}_A[0 \rightsquigarrow 1](M;\ x=0 \hookrightarrow y.N_0,\ x=1 \hookrightarrow y.N_1)$



$$\mathrm{hcom}_A[r \rightsquigarrow r](M;\ \ldots) \equiv M : A$$

$$\mathrm{hcom}_A[r \rightsquigarrow r'](M;\ \ldots,\ r_i = r_i \hookrightarrow y.N_i,\ \ldots) \equiv N_i\langle r'/y\rangle : A$$

Kan 2/2: Homogeneous Comp



Fibrant Replacement

(the cubical, syntactical way)

S1: hcom as the third constructor

Fibrant Replacement

(the cubical, syntactical way)

S1: hcom as the third constructor

add only homogeneous ones
⇨ compat with base changes
⇨ no size blow-up!

(category-theoretic argument for sizes mostly learnt from Dan Grayson)

Univalent Kan Universes

0. $V_x(A,B,E)$ type

A line between $A\langle 0/x\rangle$ and $B\langle 1/x\rangle$ witnessed by the equivalence E

1. $hcom_U[r \rightsquigarrow r'](A; \ldots)$ type

Make the universes Kan

Oh, Diagonals!

$$\mathrm{coe}_{x.\mathrm{hcom}[s \rightsquigarrow s'](A;\ \ldots)}[r \rightsquigarrow r'](M)$$

Oh, Diagonals!

$$\mathrm{coe}_{x.\mathrm{hcom}[s \rightsquigarrow s'](A;\,\ldots)}[r \rightsquigarrow r'](M)$$

$$s = s' \mapsto \mathrm{coe}_{x.A}[r \rightsquigarrow r'](M)$$

$$r = r' \mapsto M$$

Oh, Diagonals!

$$\mathrm{coe}_{x.\mathrm{hcom}[s \rightsquigarrow s'](A;\ \ldots)}[r \rightsquigarrow r'](M)$$

$$s = s' \mapsto \mathrm{coe}_{x.A}[r \rightsquigarrow r'](M)$$

$$r = r' \mapsto M$$

$$\mathrm{hcom}\ldots[s \rightsquigarrow s'](\ldots,\ r = r' \hookrightarrow \ldots)$$

diagonals for coherence conditions

Computational Semantics

A computation system for closed terms

Computational Semantics

A computation system for closed terms

$$(\lambda a.M)N \mapsto M[N/a]$$

$$(\langle x\rangle M)@r \mapsto M\langle r/x\rangle$$

$$\mathrm{coe}_{x.A}[r \rightsquigarrow r'](M) \mapsto \mathrm{coe}_{x.A'}[r \rightsquigarrow r'](M)$$

Computational Semantics

A computation system
for closed terms

$$(\lambda a.M)N \mapsto M[N/a]$$

$$(\langle x \rangle M)@r \mapsto M\langle r/x \rangle$$

$$\mathrm{coe}_{x.A}[r \rightsquigarrow r'](M) \mapsto \mathrm{coe}_{x.A'}[r \rightsquigarrow r'](M)$$

Computational semantics: values
Canonicity as a corollary

Computational Semantics

Directly usable as a type theory

$x : \mathbb{N}, y : \mathbb{N} \gg x + y \doteq y + x \in \mathbb{N}$

with all the extensionalities

See our Part III paper for details

Implementations

RedPRL

Our first try, in PRL (Nuprl) style

redprl.org

yacctt

(stay tuned for Anders' talk!)

redtt

(work in progress)

Summary of Cartesian Cubes

Canonicity + univalence?

Yes! (Part III by AFH)

And higher inductive types?

Yes! (Part IV by CH)

And the full HoTT?

Yes! (Parts III & IV)