A Mechanization of the Blakers–Massey Connectivity Theorem in Homotopy Type Theory

Favonia @ Carnegie Mellon University
Eric Finster @ LIX/École Polytechnique
Daniel R. Licata @ Wesleyan University
Peter LeFanu Lumsdaine @ Stockholm University

Homotopy Type Theory

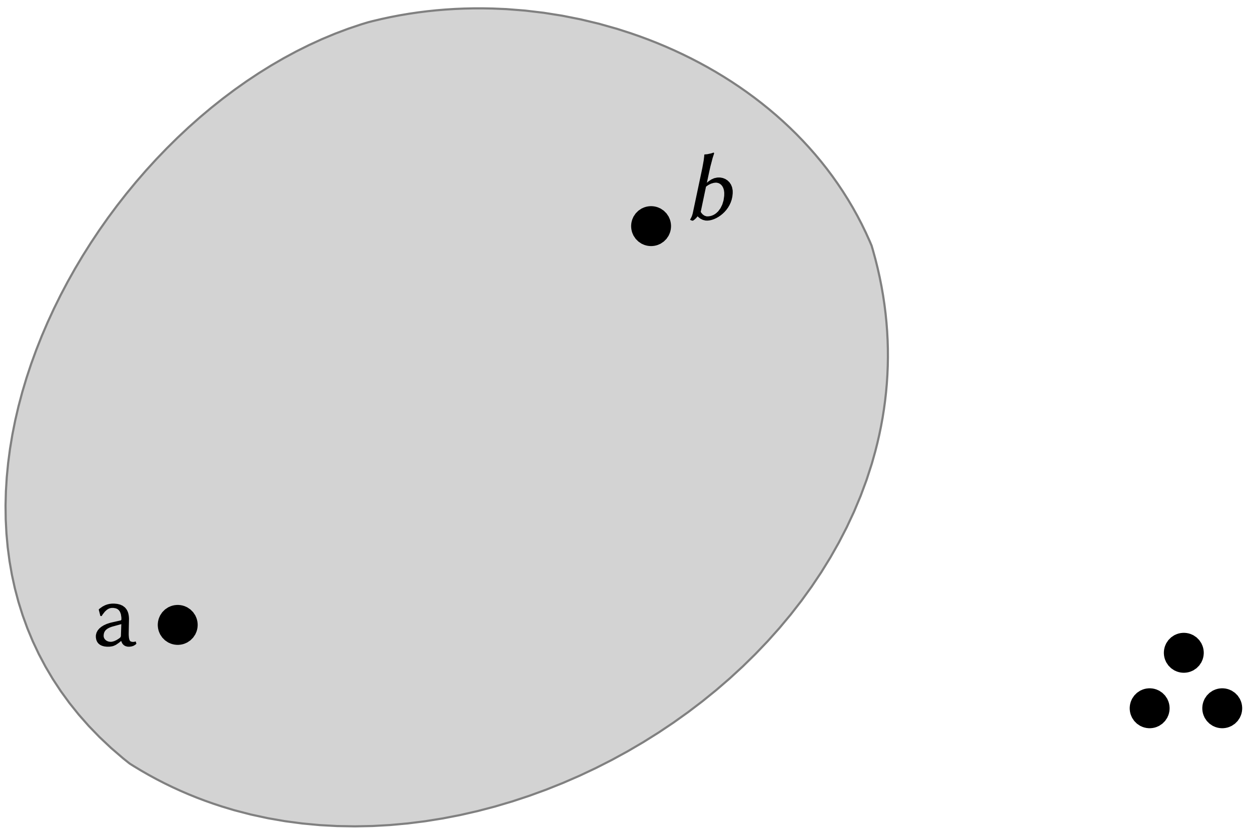
Do homotopy theory in type theory

Hopf fibrations, Eilenberg-Mac Lane spaces, van Kampen theorem [HoTT book], Mayer-Vietoris theorem [Cavallo 2014], and more…

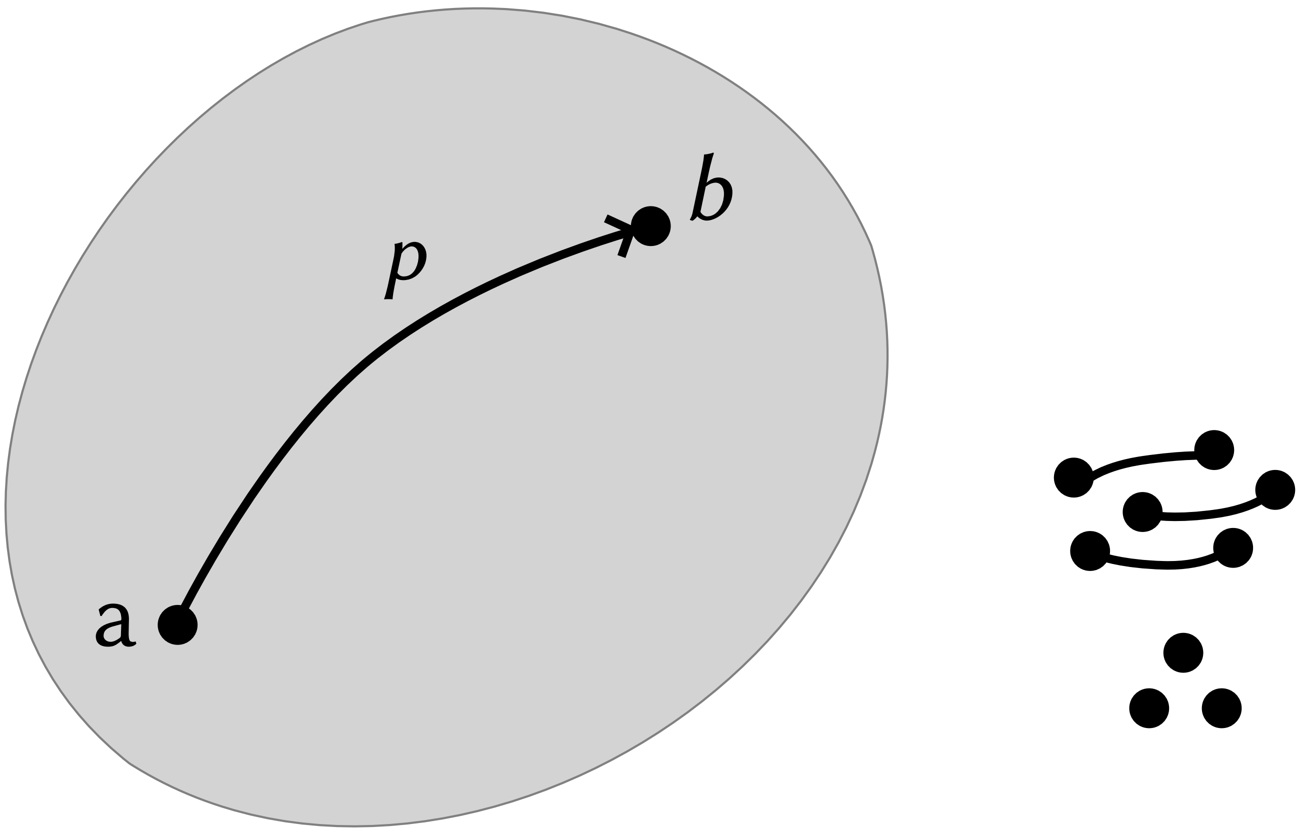
- Mechanization
- Translations to many models

↳ new research ex: in Goodwille calculus [Anel, Biedermann, Finster and Joyal 2016]

Every type is an ∞-groupoid



Every type is an ∞-groupoid



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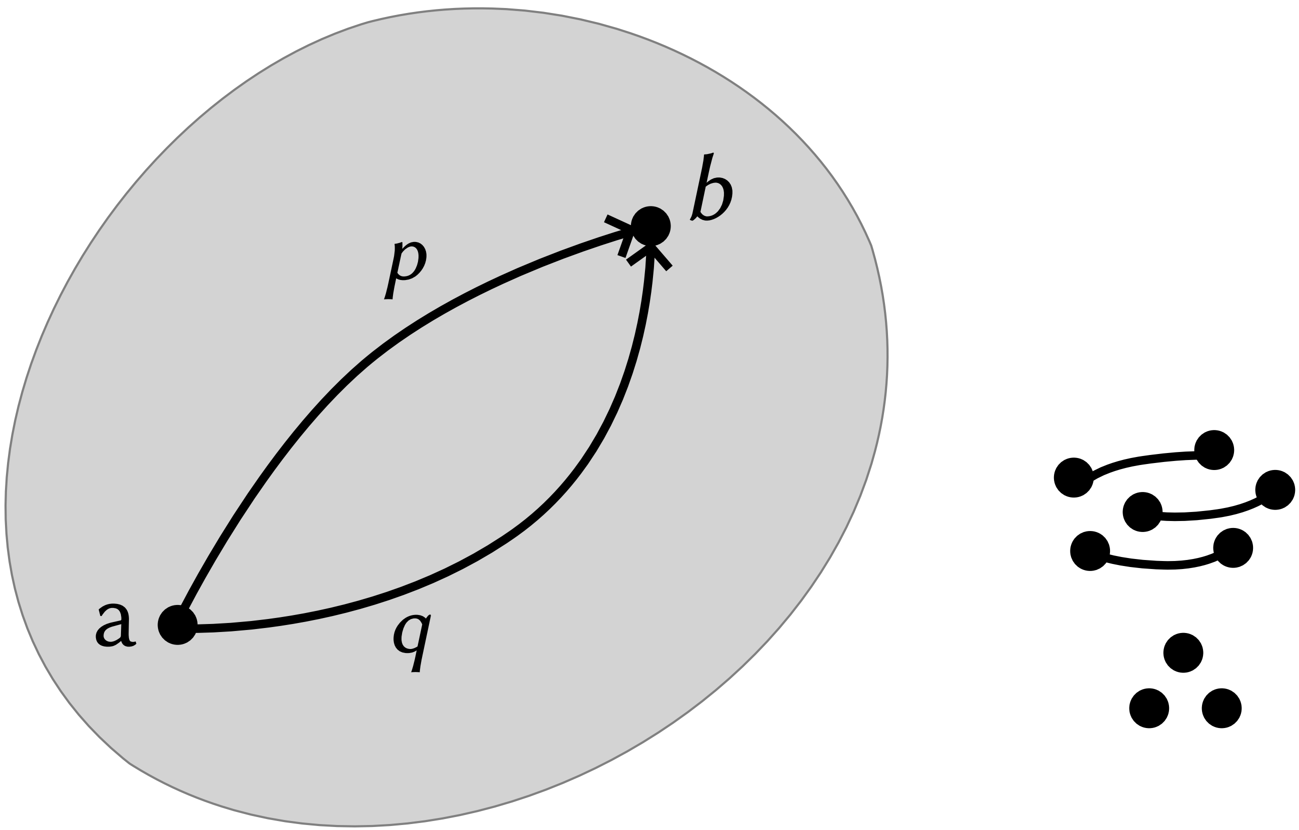

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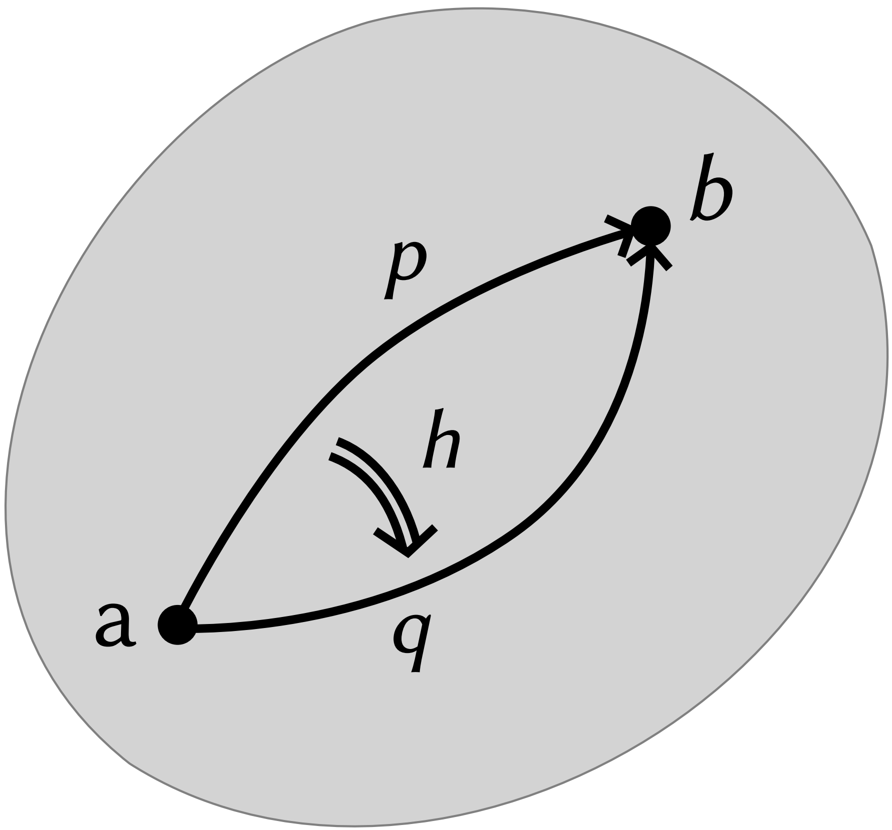
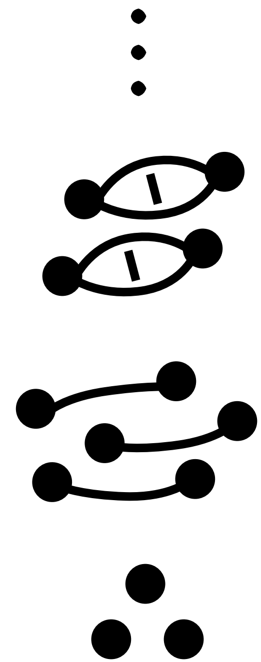
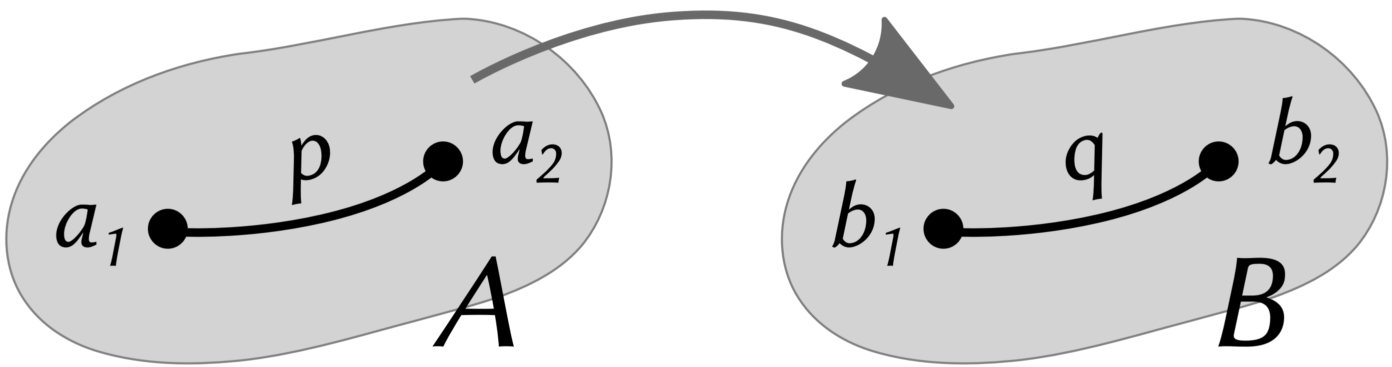




Functions preserve structures

$$f : A \longrightarrow B$$



Types and Spaces

A	Type	Space
$a : A$	Term	Point
$f : A \longrightarrow B$	Function	Continuous Mapping
$C : A \longrightarrow Type$	Dependent Type	Fibration
$C(a)$		Fiber
$p : a =_A b$	Identification	Path

Blakers-Massey is for calculating higher homotopy groups of pushouts

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mappings from spheres
to the space

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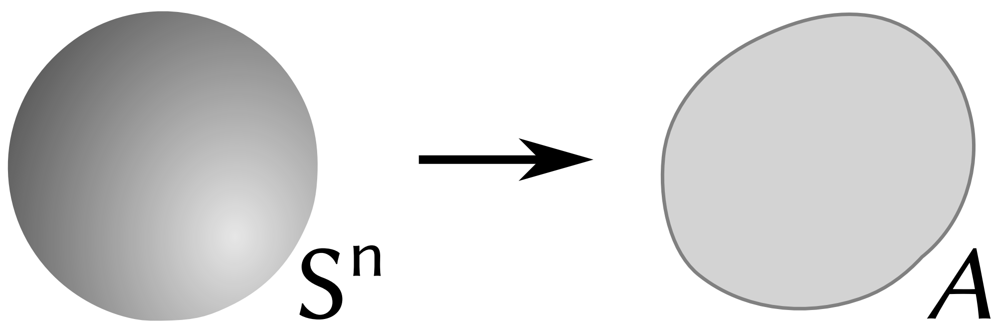
mappings from spheres to the space

two spaces glued together

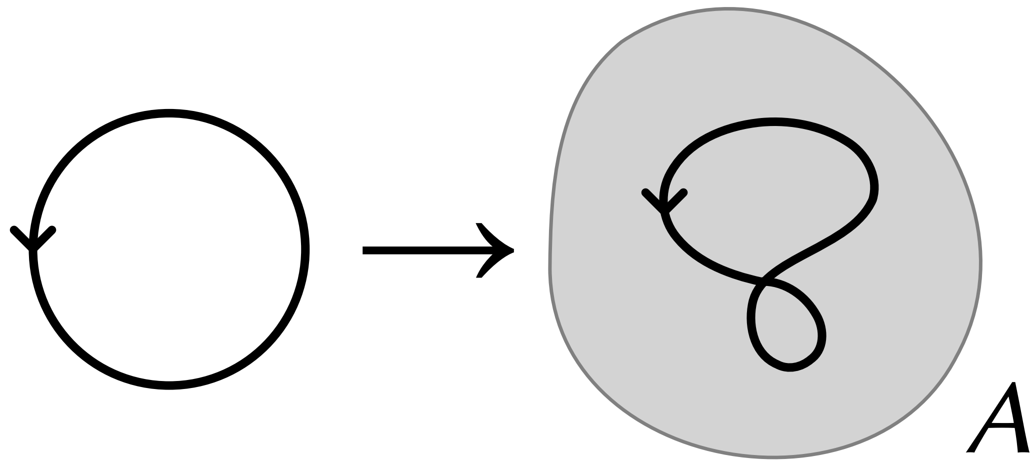
Homotopy Groups

{ mappings from the *n*-sphere }



"higher" if $n > 1$

First Homotopy Group

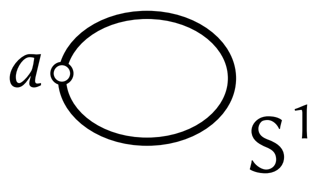


Mappings from the circle to A
= Images of the circle in A
= (Directed) loops in A

First Homotopy Group

(= directed loops in the space)

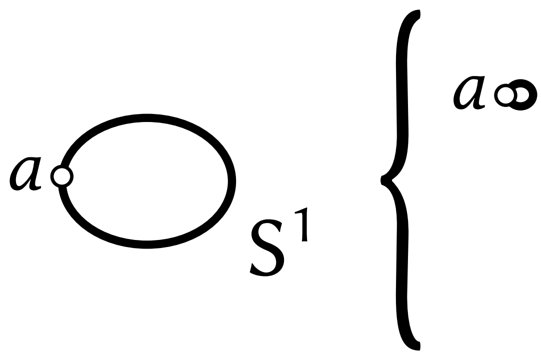
How many ways to go from a to a?



First Homotopy Group

(= directed loops in the space)

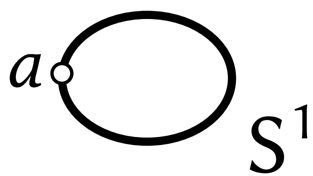
How many ways to go from *a* to *a*?

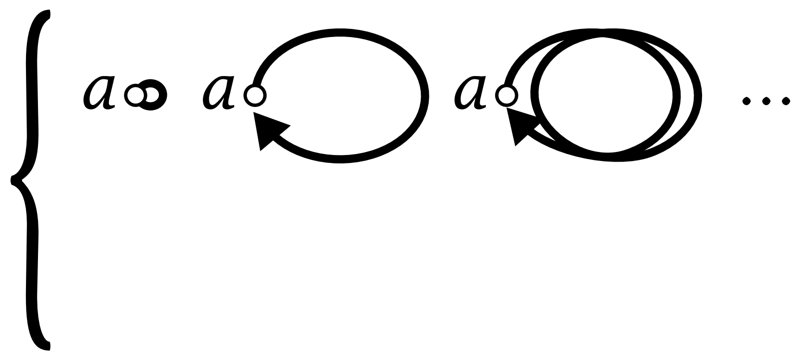


First Homotopy Group

(= directed loops in the space)

How many ways to go from *a* to *a*?

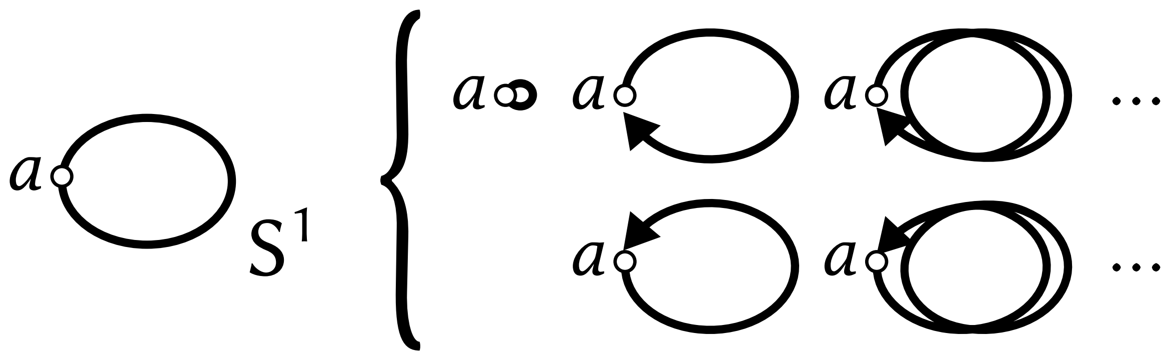




First Homotopy Group

(= directed loops in the space)

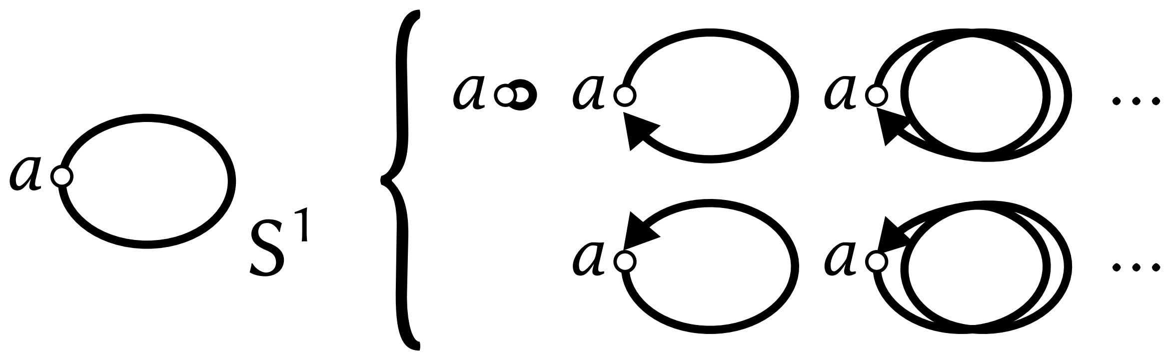
How many ways to go from *a* to *a*?

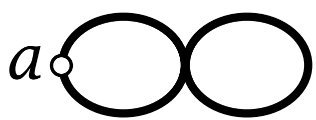


First Homotopy Group

(= directed loops in the space)

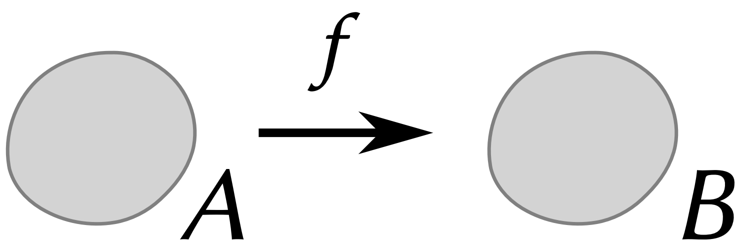
How many ways to go from *a* to *a*?





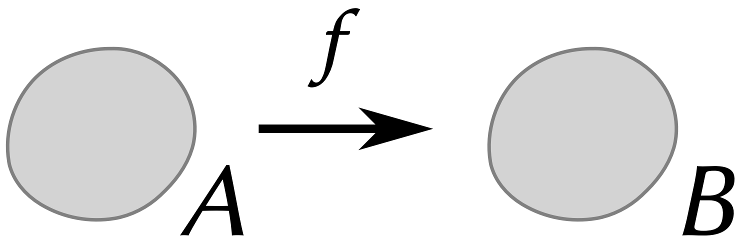
(much more)

Connectivity



Connectivity

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ \pi_2(A) & \longrightarrow & \pi_2(B) \\ \pi_1(A) & \longrightarrow & \pi_1(B) \end{array}$$

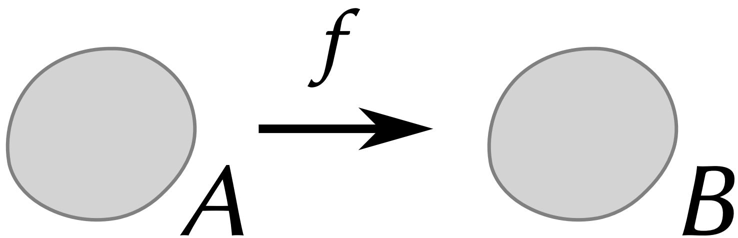


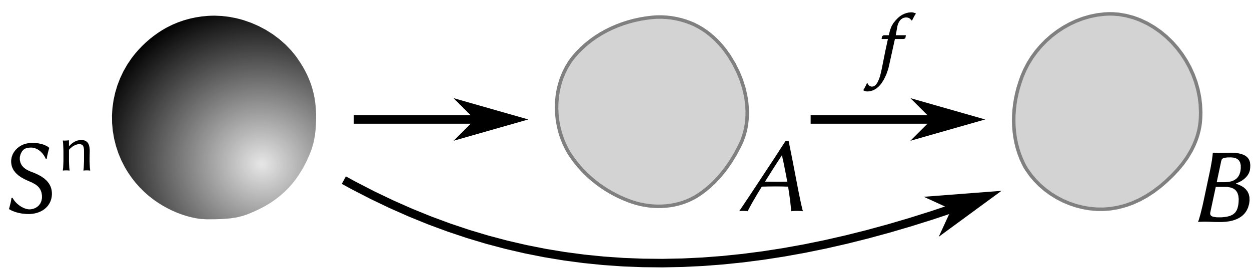
Connectivity

$$\vdots \qquad \vdots \qquad \vdots$$

$$\pi_2(A) \longrightarrow \pi_2(B)$$

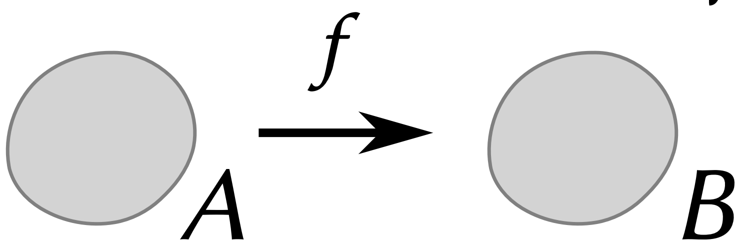
$$\pi_1(A) \longrightarrow \pi_1(B)$$





Connectivity

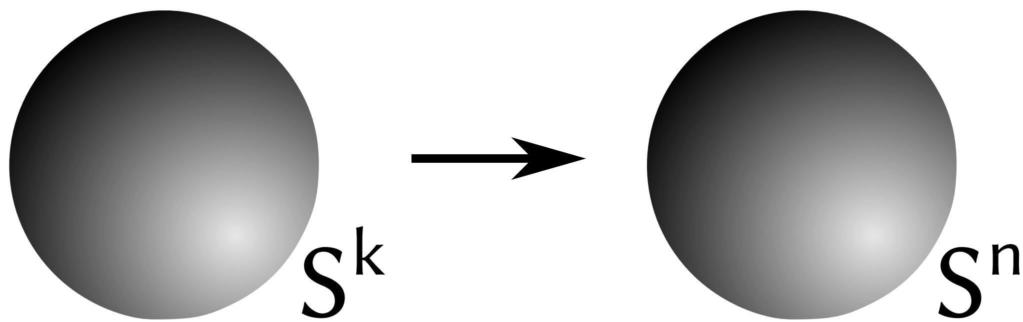
$$\left.\begin{array}{ccc} \vdots & \vdots & \vdots \\ \pi_2(A) & \xrightarrow{\sim} & \pi_2(B) \\ \pi_1(A) & \xrightarrow{\sim} & \pi_1(B) \end{array}\right\}$$ first n pairs are isomorphic



f is n-connected

if inducing isomorphisms up to n

Homotopy Groups of Spheres



Homotopy Groups of Spheres

	1	2	3	4	5	6	7	8	9	10
S^1	Z	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^4	0	0	0	Z	Z_2	Z_2	$Z \times Z_{12}$	Z_2^2	Z_2^2	$Z_{24} \times Z_3$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0

Homotopy Groups of Spheres

	1	2	3	4	5	6	7	8	9	10
S^1	Z	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^4	0	0	0	Z	Z_2	Z_2	$Z \times Z_{12}$	Z_2^2	Z_2^2	$Z_{24} \times Z_3$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0

Homotopy Groups of Spheres

	1	2	3	4	5	6	7	8	9	10
S^1	Z	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^4	0	0	0	Z	Z_2	Z_2	$Z \times Z_{12}$	Z_2^2	Z_2^2	$Z_{24} \times Z_3$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0

Homotopy Groups of Spheres

	1	2	3	4	5	6	7	8	9	10
S^1	Z	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^4	0	0	0	Z	Z_2	Z_2	$Z \times Z_{12}$	Z_2^2	Z_2^2	$Z_{24} \times Z_3$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0

Homotopy Groups of Spheres

	1	2	3	4	5	6	7	8	9	10
S^1	Z	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^4	0	0	0	Z	Z_2	Z_2	$Z \times Z_{12}$	$Z_2^{\,2}$	$Z_2^{\,2}$	$Z_{24} \times Z_3$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0

Homotopy Groups of Spheres

	1	2	3	4	5	6	7	8	9	10
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S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^4	0	0	0	Z	Z_2	Z_2	$Z \times Z_{12}$	Z_2^2	Z_2^2	$Z_{24} \times Z_3$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0

Homotopy Groups of Spheres

	1	2	3	4	5	6	7	8	9	10
S^1	Z	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^4	0	0	0	Z	Z_2	Z_2	$Z \times Z_{12}$	Z_2^2	Z_2^2	$Z_{24} \times Z_3$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0

Homotopy Groups of Spheres

	1	2	3	4	5	6	7	8	9	10
S^1	Z	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^4	0	0	0	Z	Z_2	Z_2	$Z \times Z_{12}$	Z_2^2	Z_2^2	$Z_{24} \times Z_3$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0

Blakers-Massey is for calculating
higher homotopy groups of pushouts

mappings from spheres
to the space

two spaces
glued together

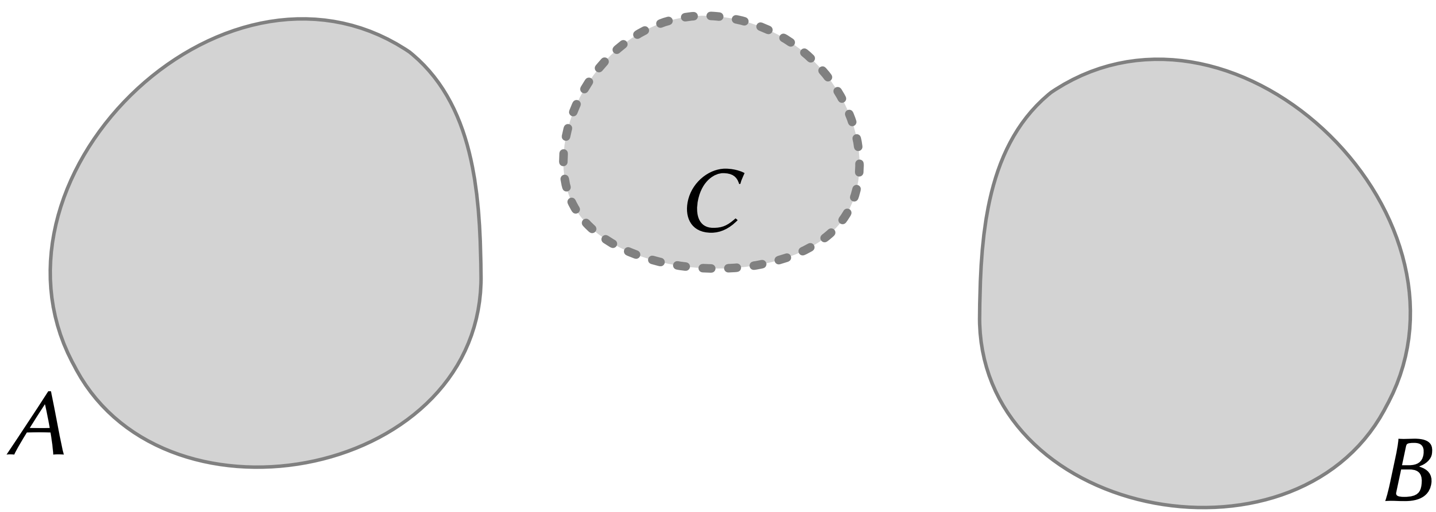
We will show homotopy groups
of spheres eventually stabilize

Pushouts

Disjoint sums with gluing

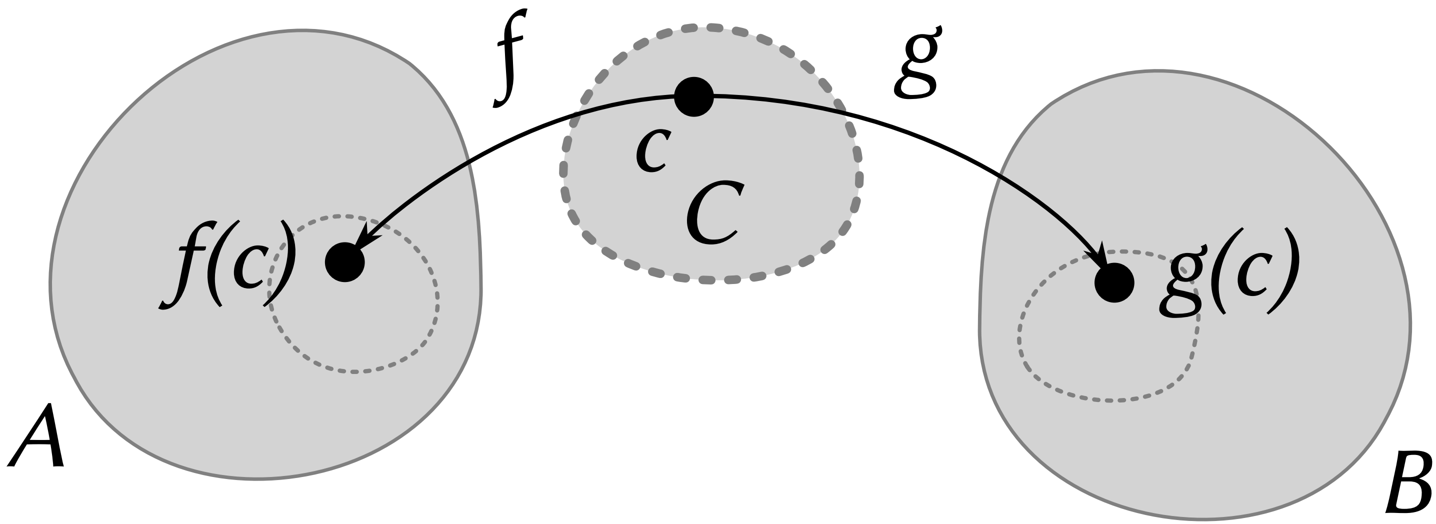
Pushouts

Disjoint sums with gluing



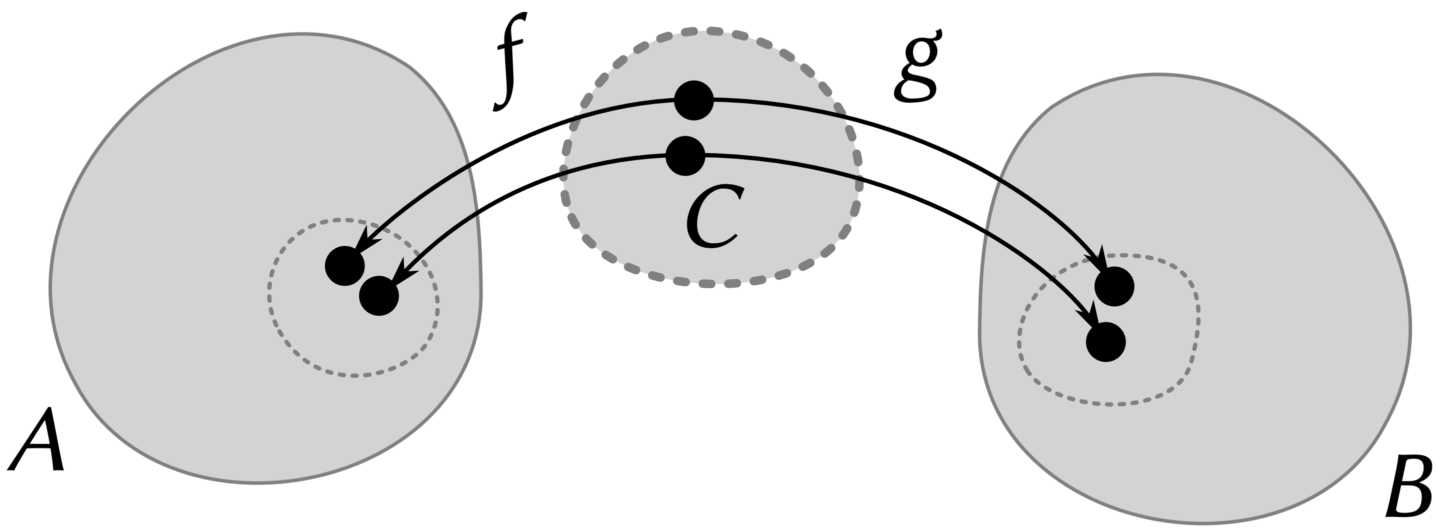
Pushouts

Disjoint sums with gluing



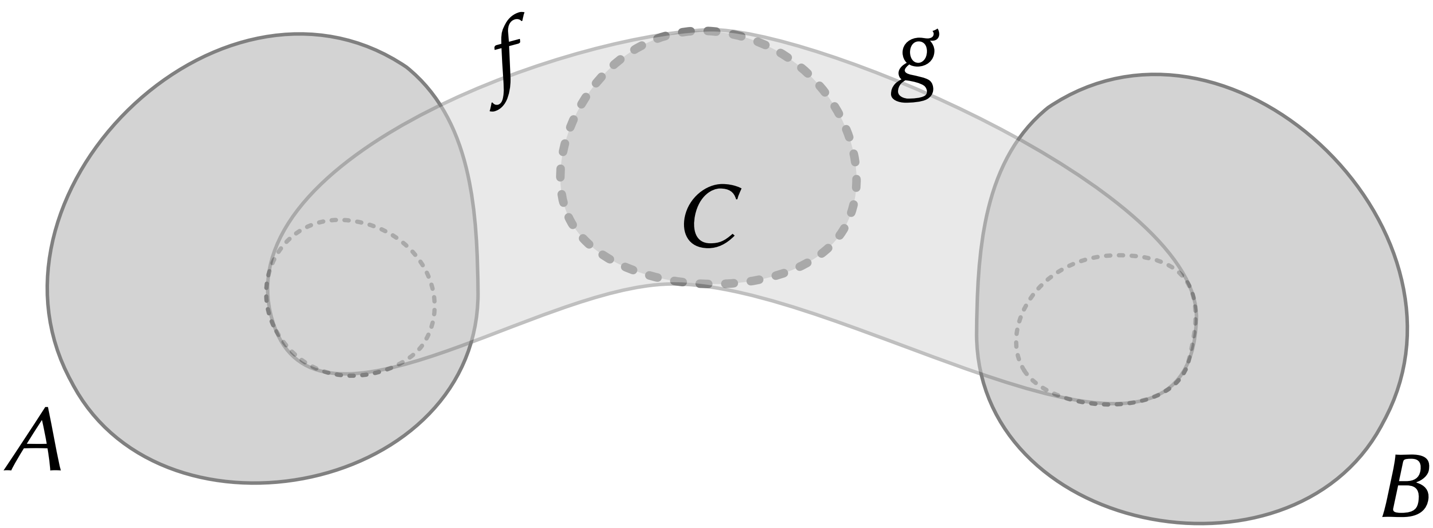
Pushouts

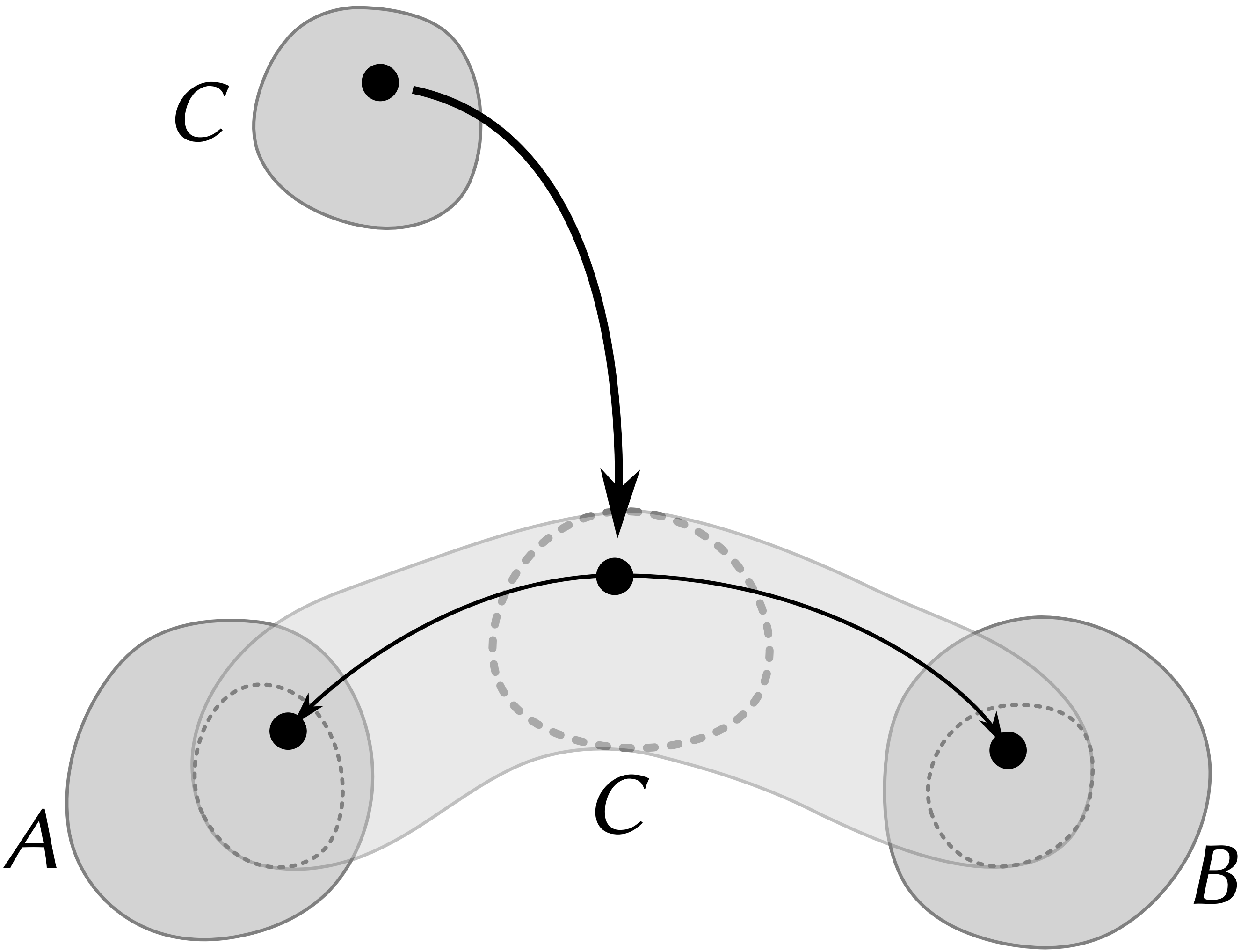
Disjoint sums with gluing

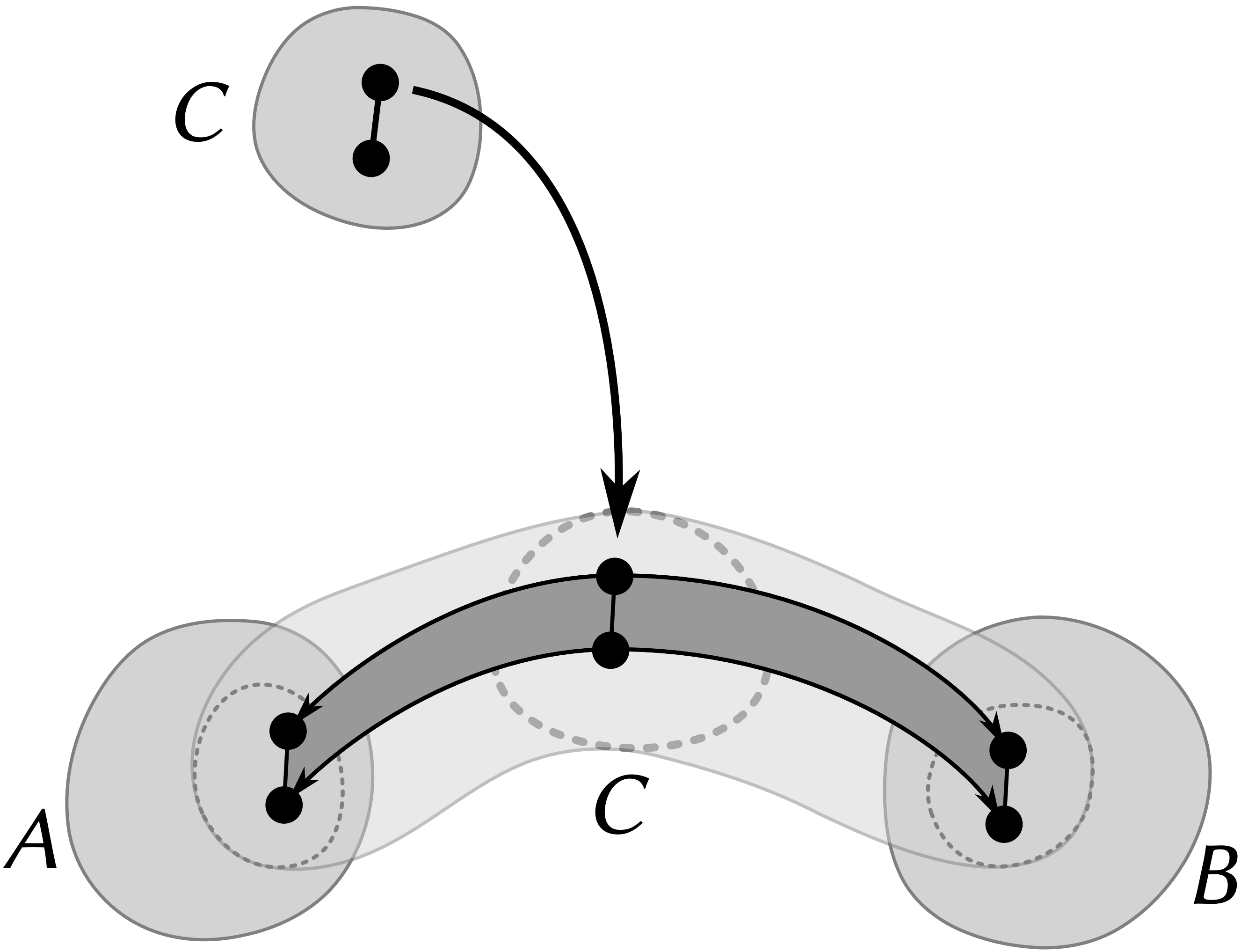


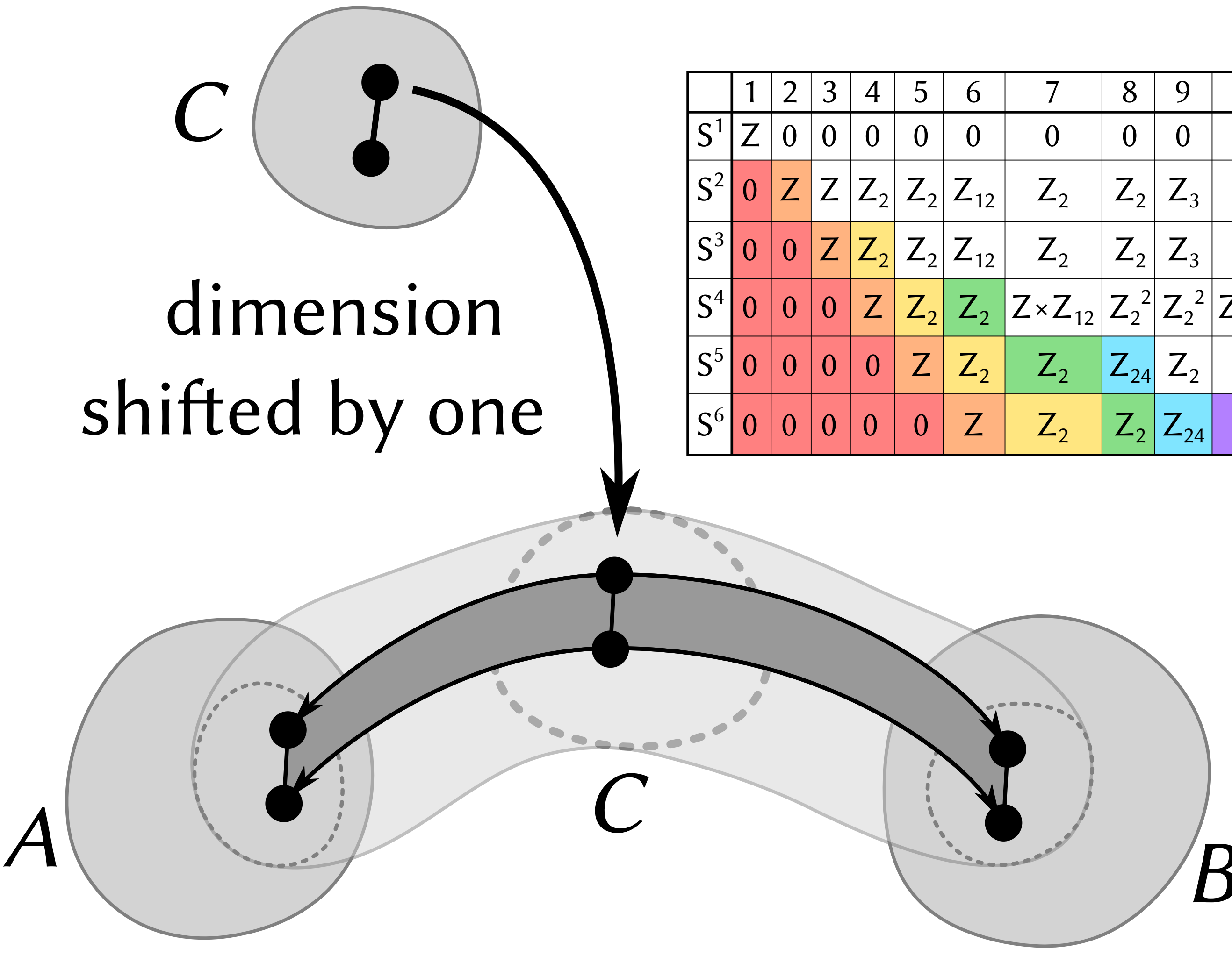
Pushouts

Disjoint sums with gluing



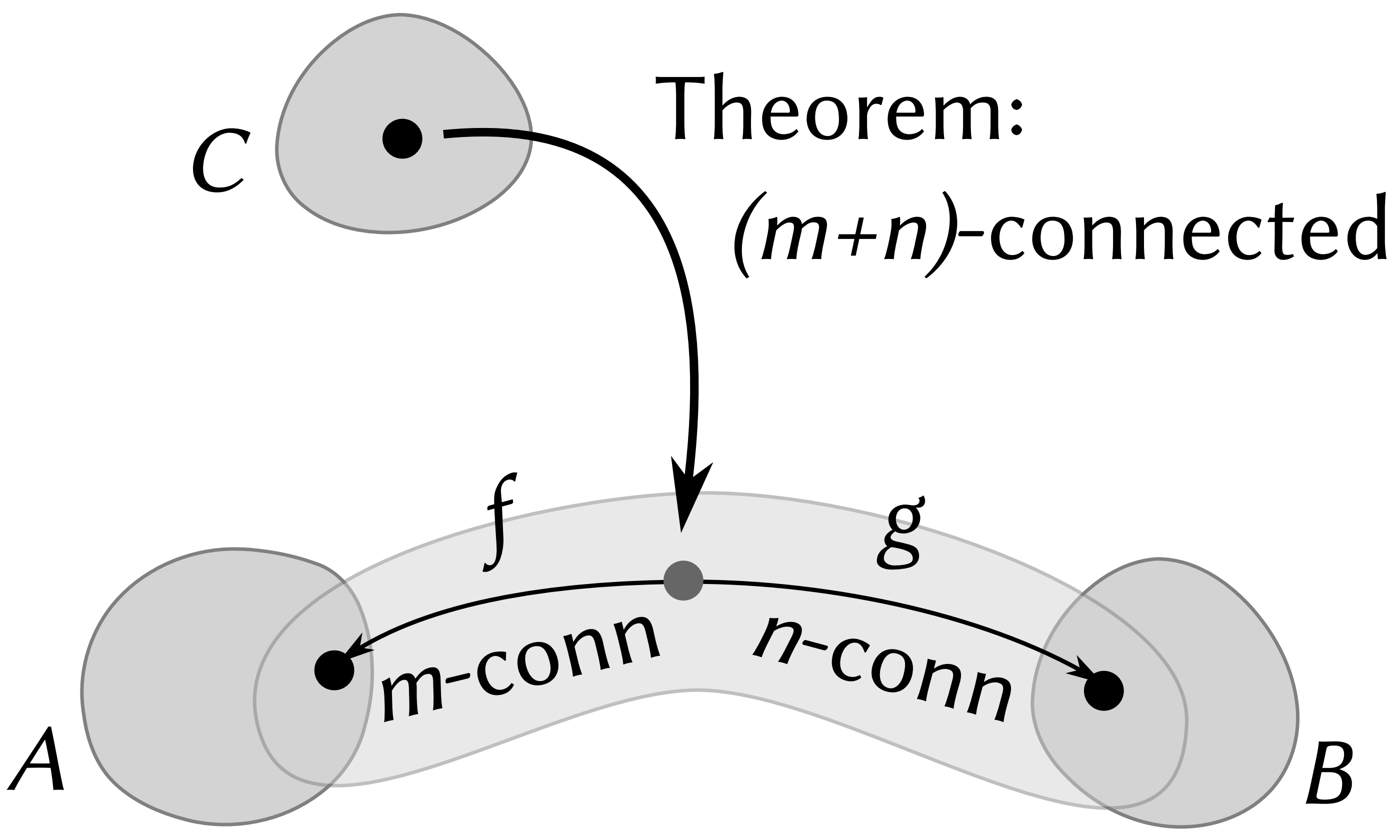

C
C
A
B


C
C
A
B



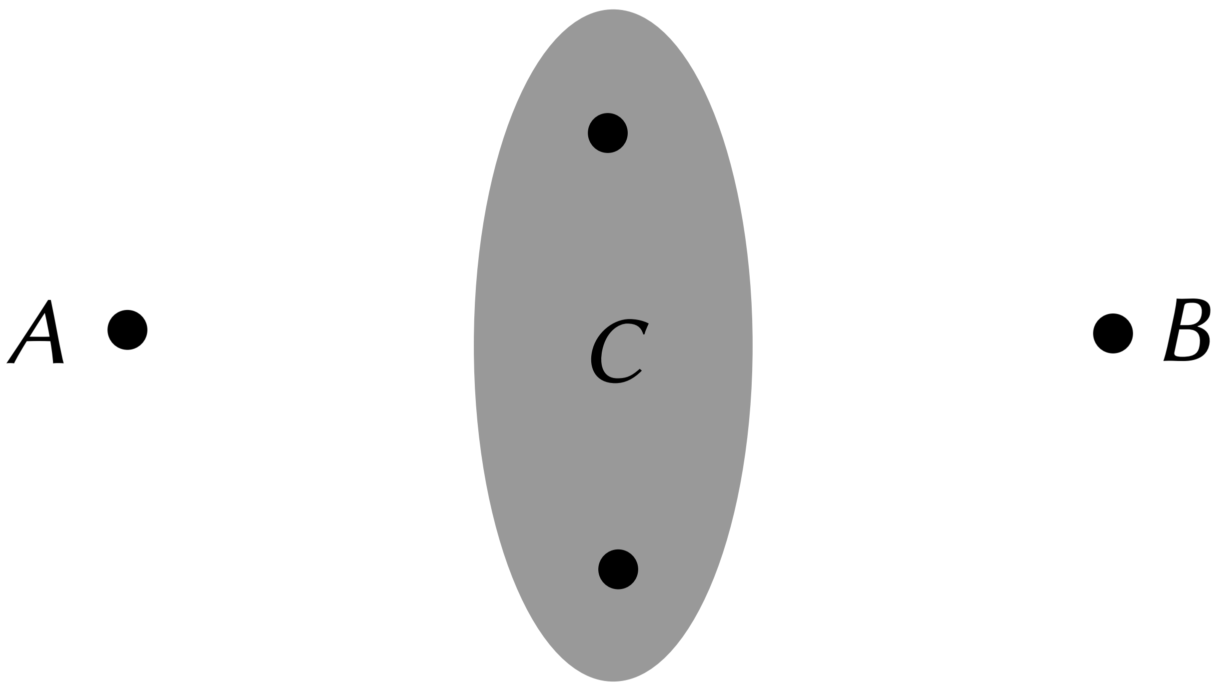
	1	2	3	4	5	6	7	8	9	10
S^1	Z	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^4	0	0	0	Z	Z_2	Z_2	$Z \times Z_{12}$	$Z_2{}^2$	$Z_2{}^2$	$Z_{24} \times Z_3$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0

Blakers-Massey Theorem



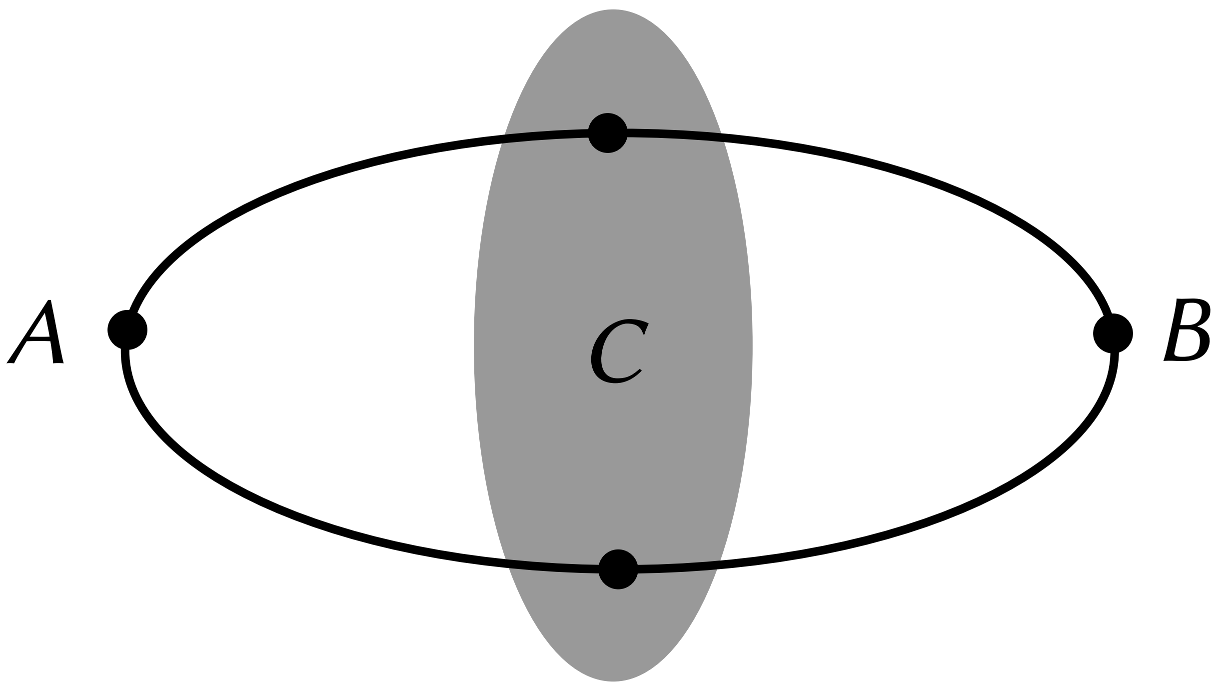
Spheres as Pushouts

1-sphere (circle)



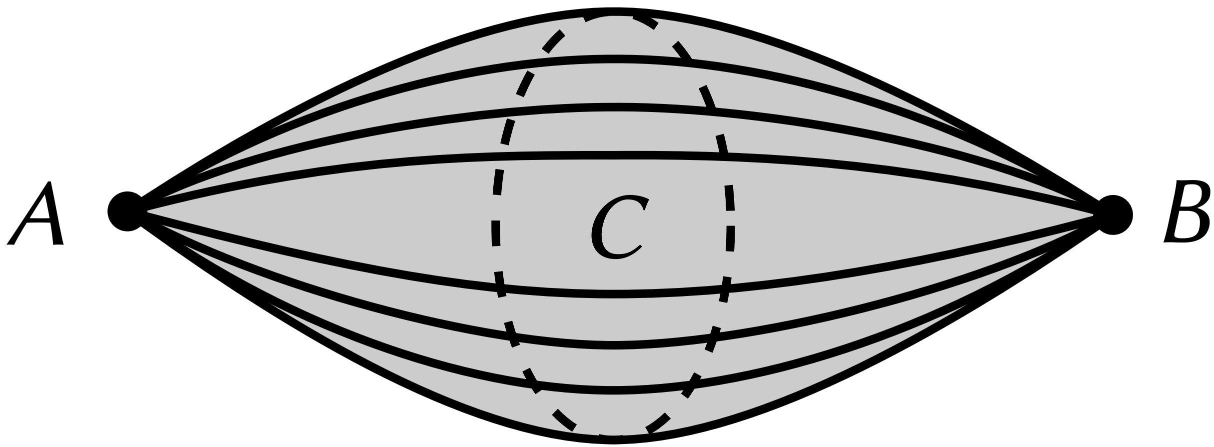
Spheres as Pushouts

1-sphere (circle)



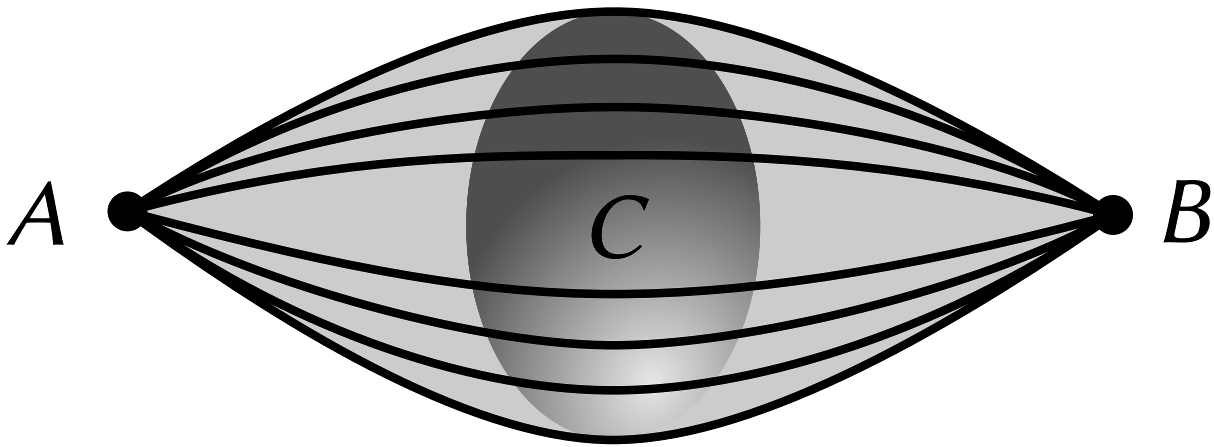
Spheres as Pushouts

2-sphere

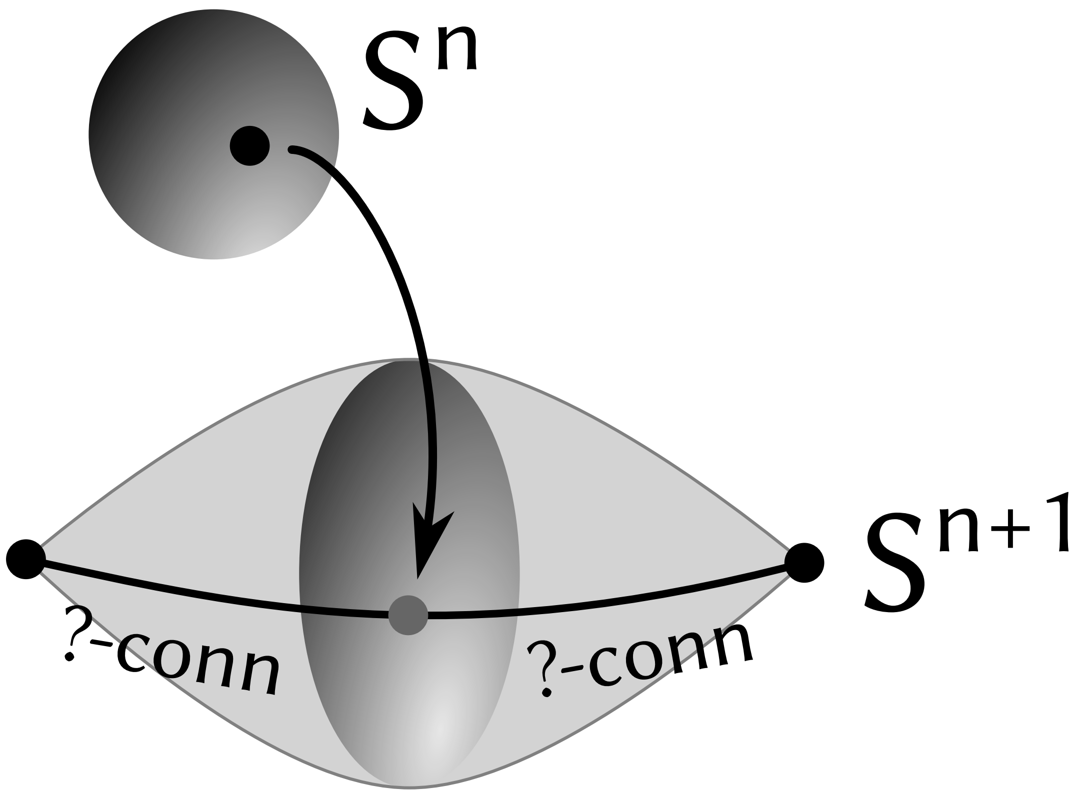


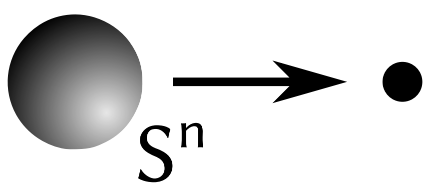
Spheres as Pushouts

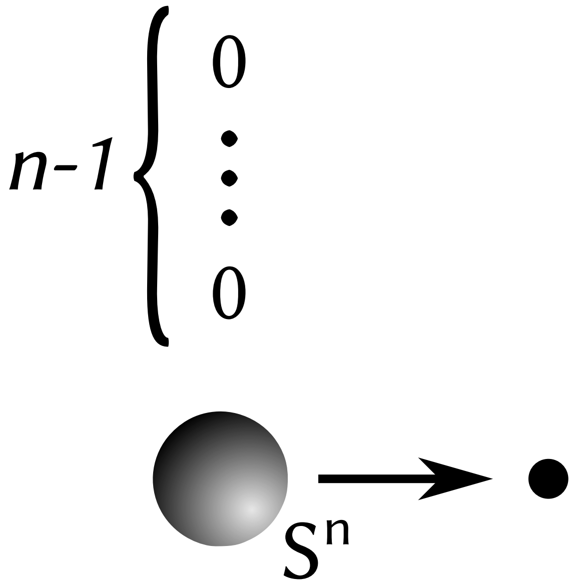
(n+1)-sphere from *n*-sphere



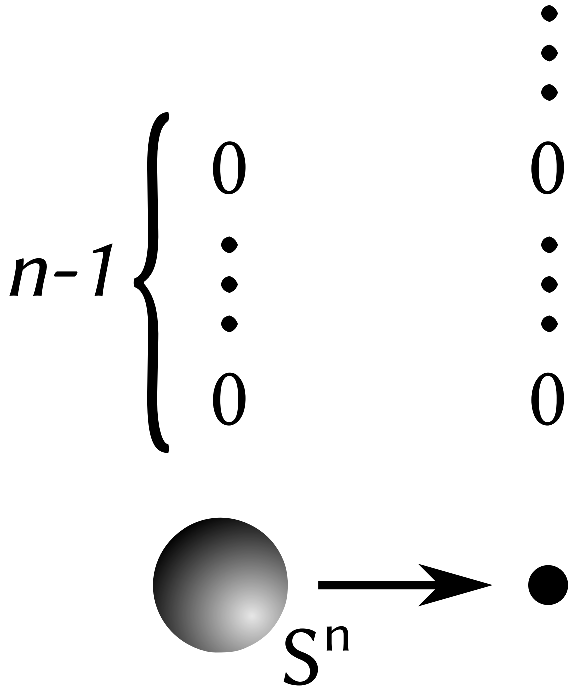
Blakers-Massey on Spheres



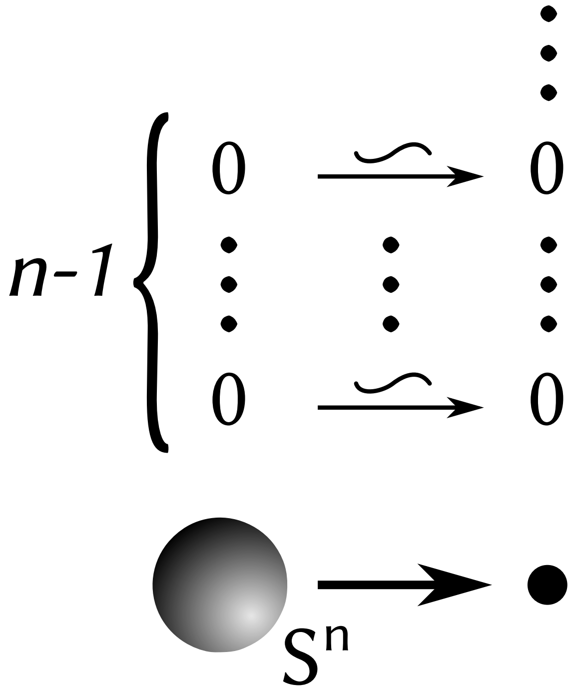

S^n



	1	2	3	4	5	6
S^1	Z	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}
S^3	0	0	Z	Z_2	Z_2	Z_{12}
S^4	0	0	0	Z	Z_2	Z_2
S^5	0	0	0	0	Z	Z_2
S^6	0	0	0	0	0	Z



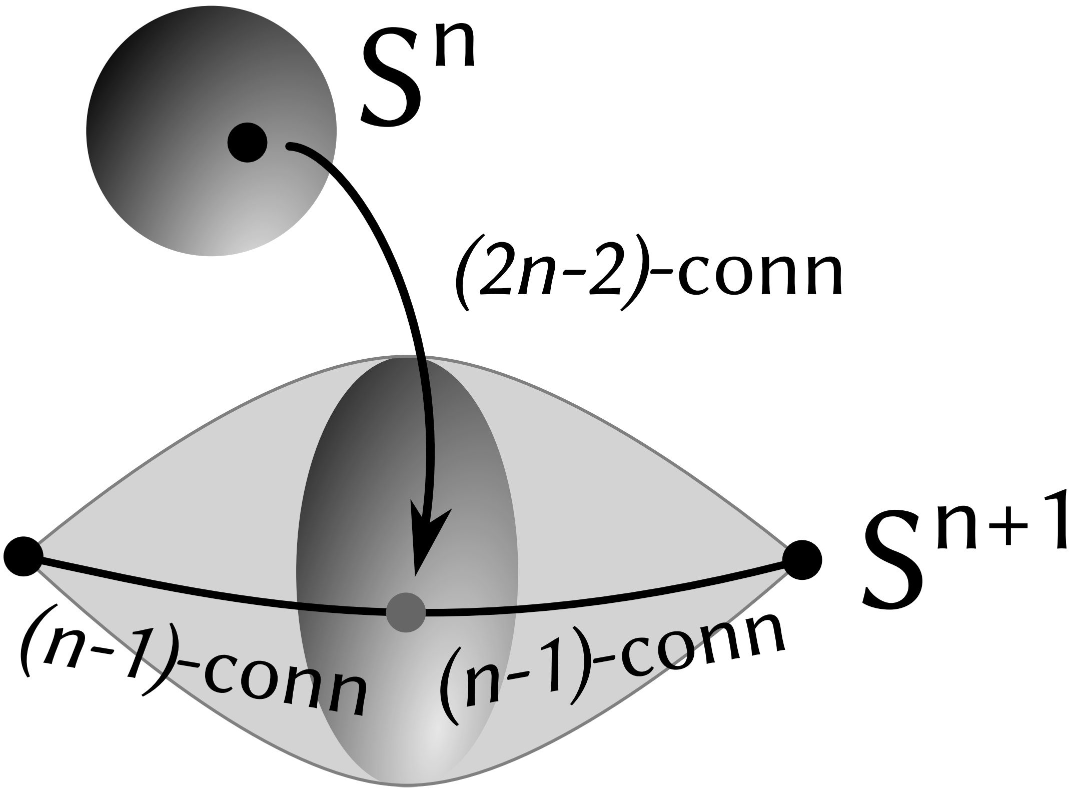
	1	2	3	4	5	6
S^1	Z	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}
S^3	0	0	Z	Z_2	Z_2	Z_{12}
S^4	0	0	0	Z	Z_2	Z_2
S^5	0	0	0	0	Z	Z_2
S^6	0	0	0	0	0	Z



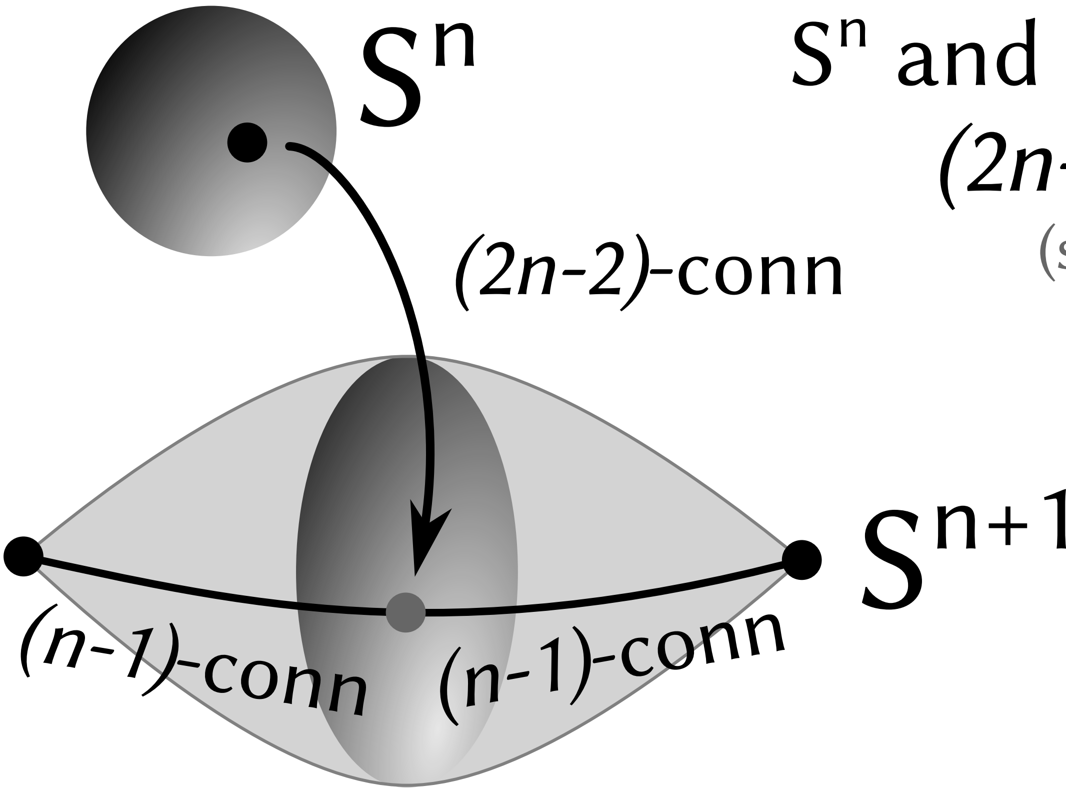
(n-1)-connected!

	1	2	3	4	5	6
S^1	Z	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}
S^3	0	0	Z	Z_2	Z_2	Z_{12}
S^4	0	0	0	Z	Z_2	Z_2
S^5	0	0	0	0	Z	Z_2
S^6	0	0	0	0	0	Z

Blakers-Massey on Spheres



Blakers-Massey on Spheres



S^n and S^{n+1} share
(2n-2) groups
(shifted by one)

Homotopy Groups of Spheres

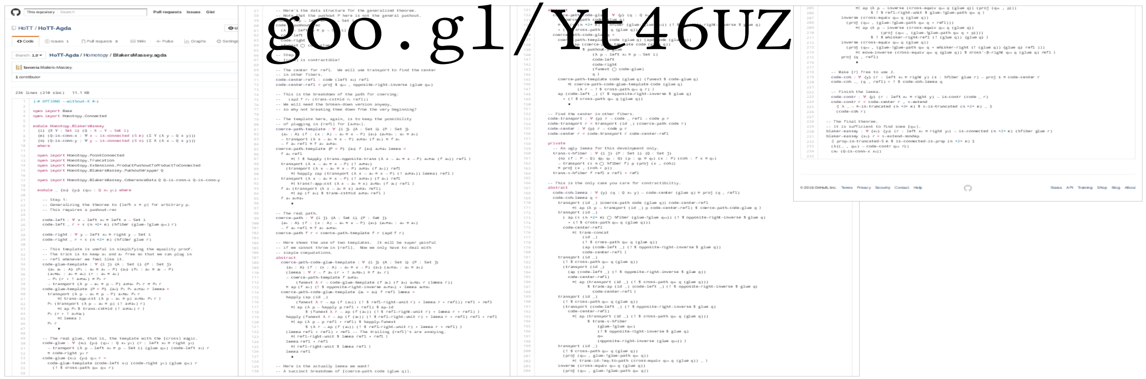
	1	2	3	4	5	6	7	8	9	10
S^1	Z	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^4	0	0	0	Z	Z_2	Z_2	$Z \times Z_{12}$	Z_2^2	Z_2^2	$Z_{24} \times Z_3$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0

Homotopy Groups of Spheres

	1	2	3	4	5	6	7	8	9	10
S^1	Z	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}
S^4	0	0	0	Z	Z_2	Z_2	$Z \times Z_{12}$	$Z_2{}^2$	$Z_2{}^2$	$Z_{24} \times Z_3$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0

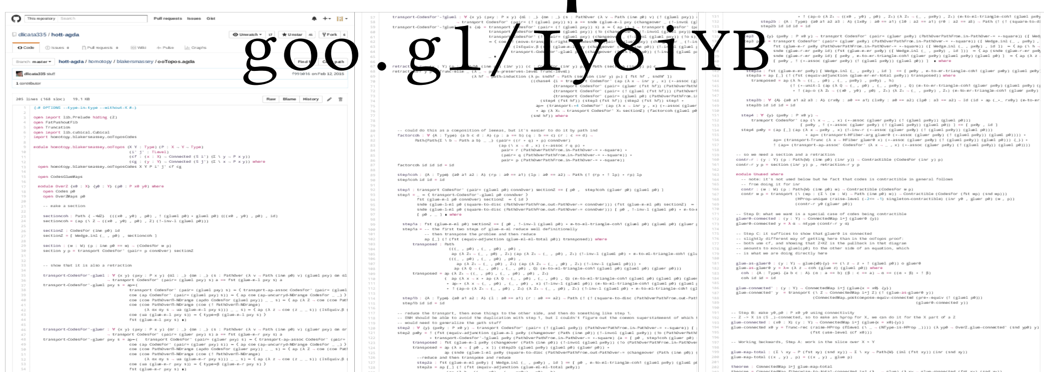
Two Mechanized Proofs

one of direct style



goo.gl/Yt46UZ

one with ∞-topos in mind

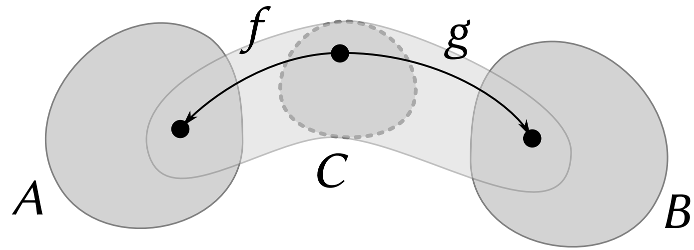


goo.gl/Iy8iYB

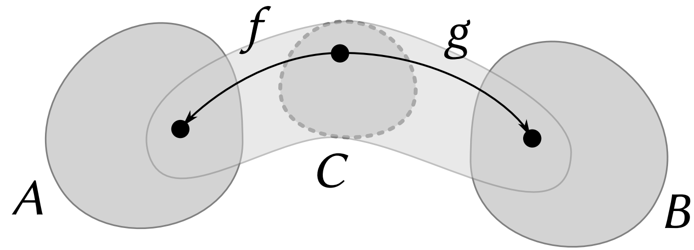
Conclusion

A new proof of Blakers-Massey which is mechanized in Agda and leads to new math research

See our paper for more details!




```
data Pushout (A B C : Type)
             (f : C → A)
             (g : C → B) : Type where
  left  : A → Pushout A B C f g
  right : B → Pushout A B C f g
  glue  : (c : C) → left (f c) == right (g c)
```




```
data Pushout (A B C : Type)
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  glue  : (c : C) → left (f c) == right (g c)
```

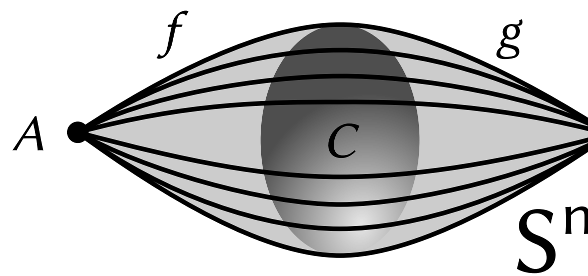


```
Sphere : ℕ → Type
Sphere 0     = Bool
Sphere (S n) =
  Pushout Unit Unit (Sphere n) (λ _ → tt) (λ _ → tt)
```

A B C f g