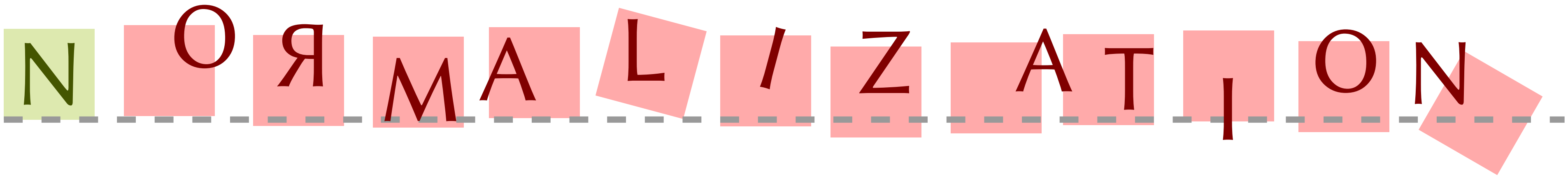
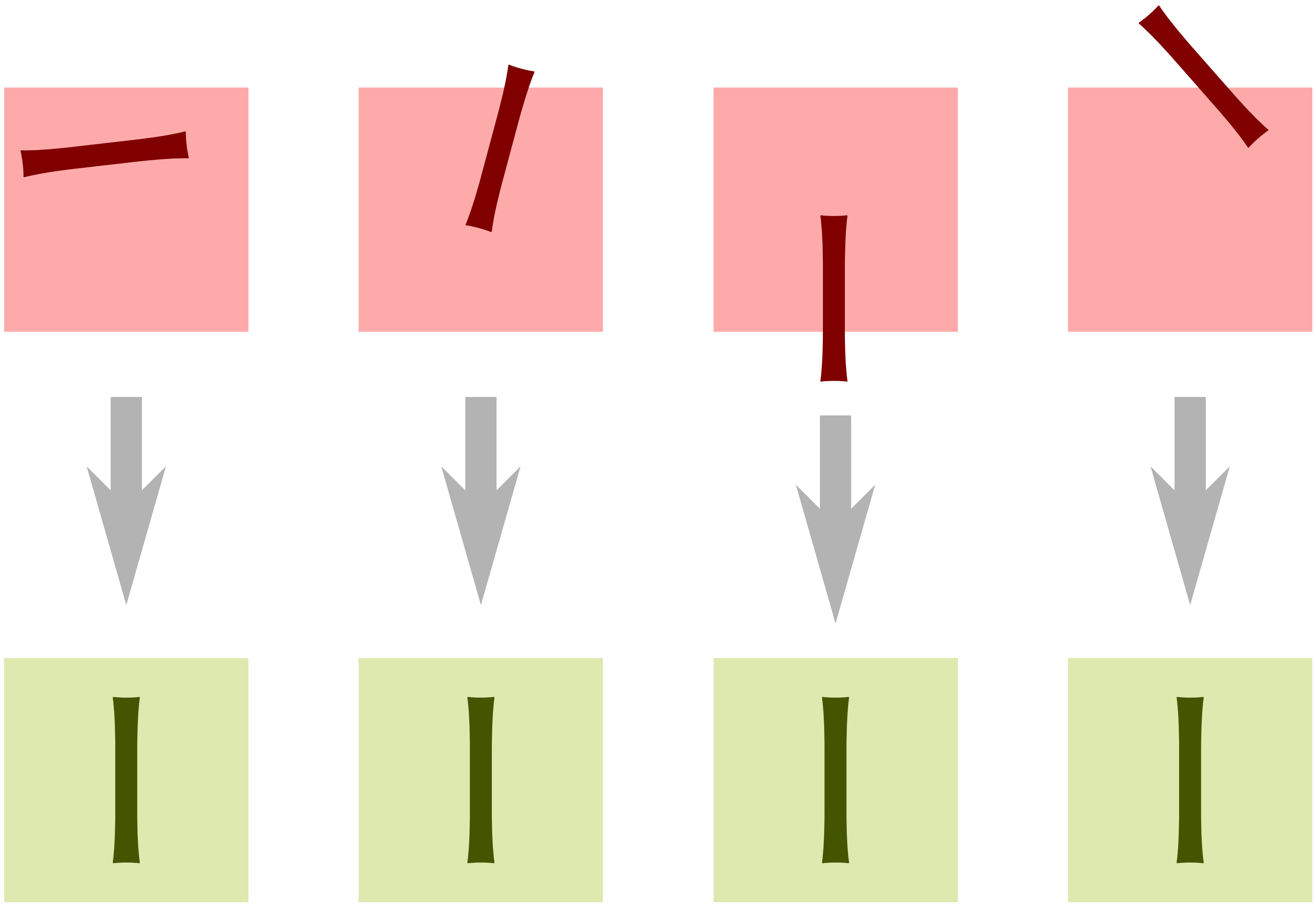
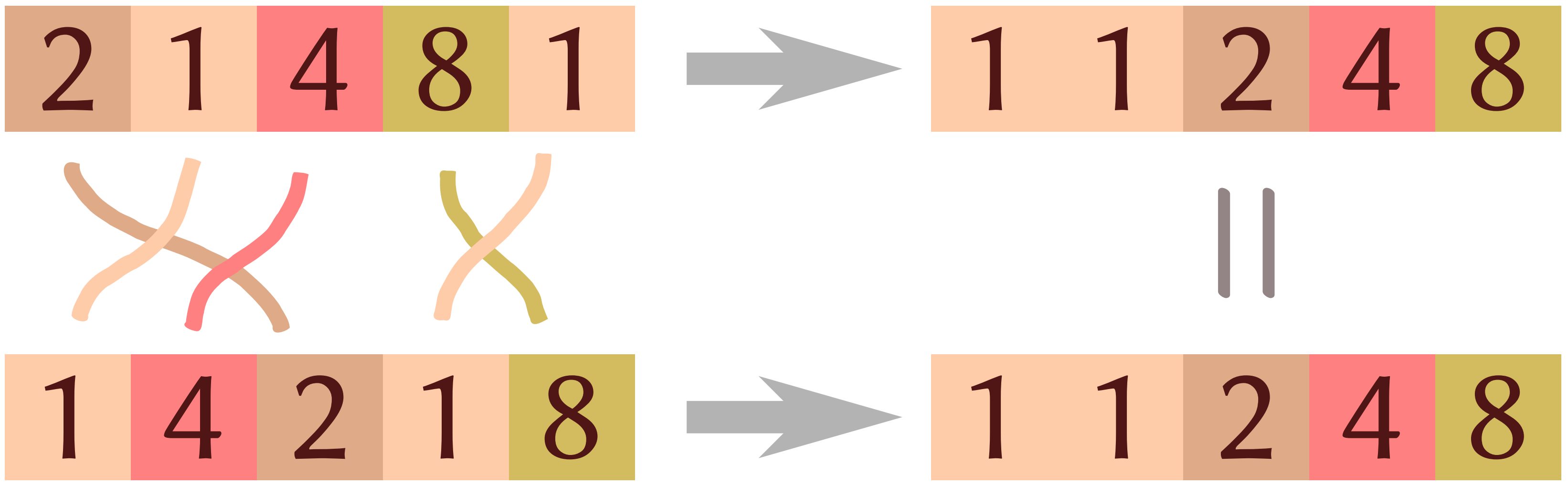
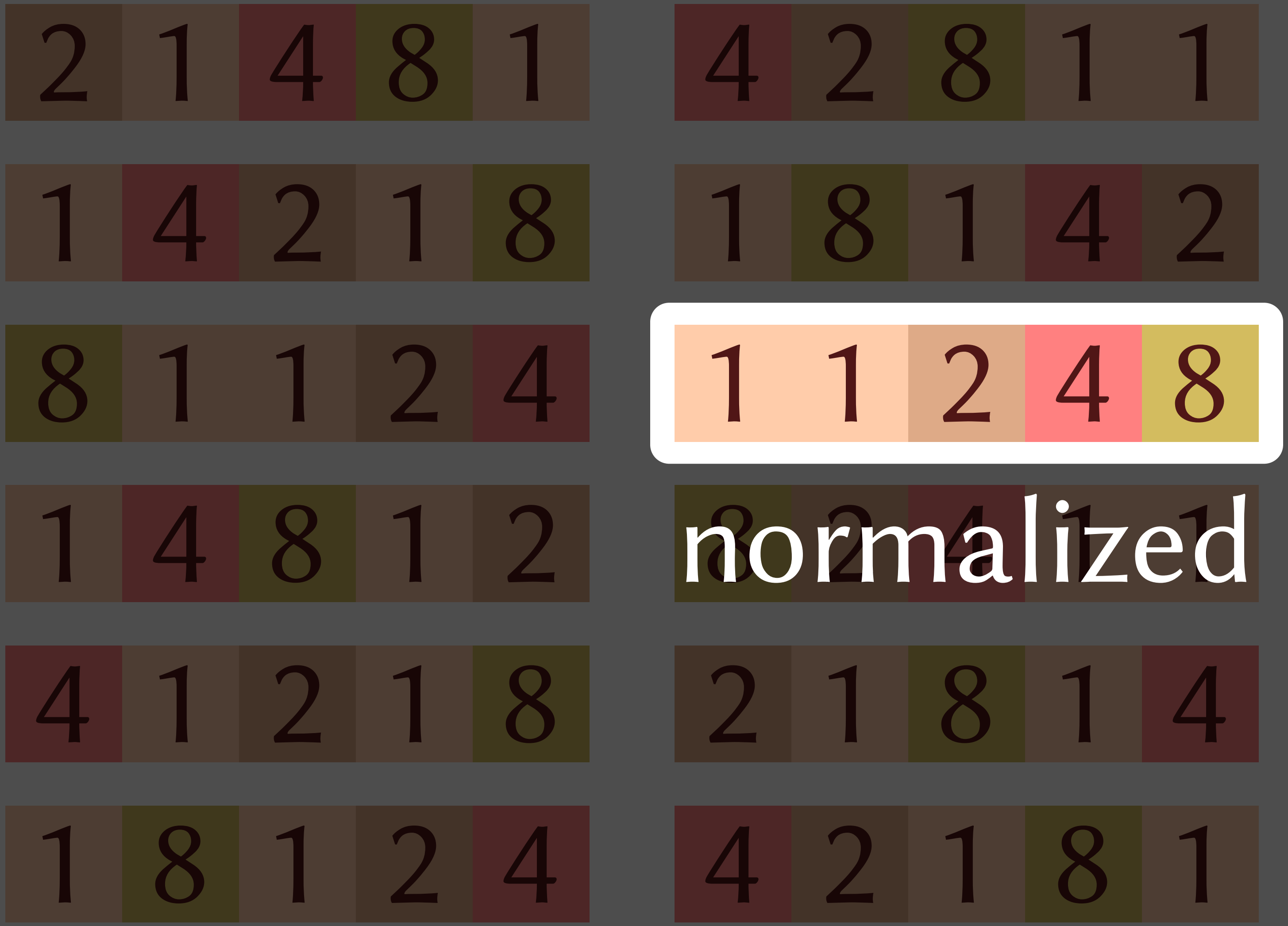
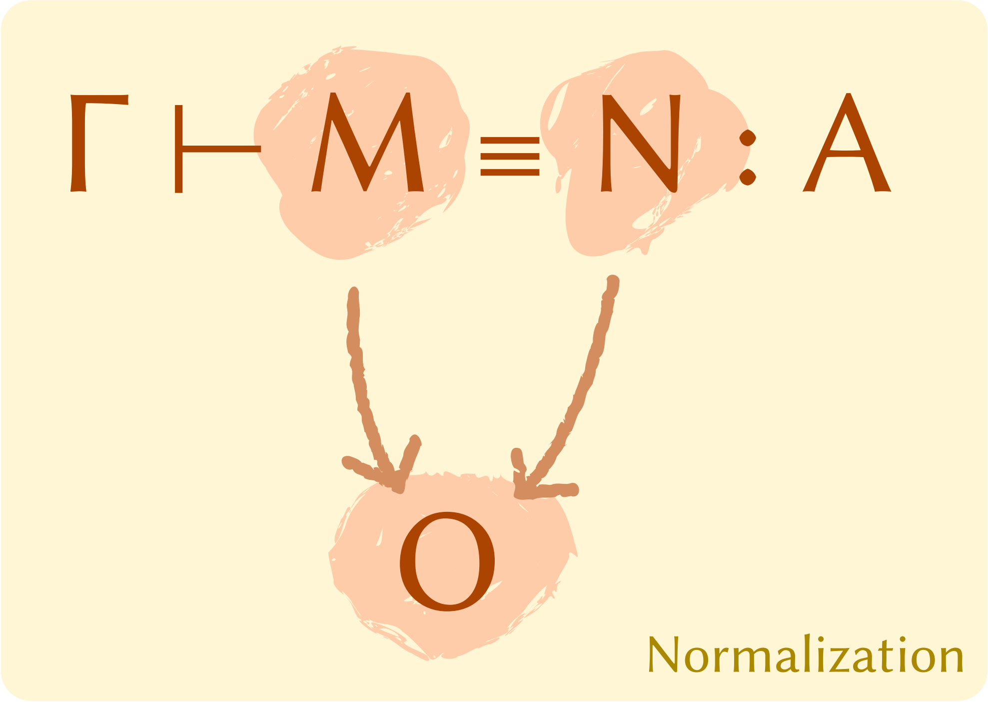

NORMALIZATION




2 1 4 8 1
1 1 2 4 8
1 4 2 1 8
1 1 2 4 8


2 1 4 8 1
4 2 8 1 1
1 4 2 1 8
1 8 1 4 2
8 1 1 2 4
1 1 2 4 8
1 4 8 1 2
normalized
4 1 2 1 8
2 1 8 1 4
1 8 1 2 4
4 2 1 8 1



Consistency
Completeness
Type-checking
Unification
(…many others)

x $\qquad$ $\lambda x.M \quad M(N) \quad A \longrightarrow B$

$\langle M, N\rangle \quad \mathsf{fst}(M) \quad \mathsf{snd}(M) \quad A \times B$

$\mathsf{inl}(M) \quad \mathsf{inr}(M) \quad \mathsf{case}(x.M;\, y.N;\, O) \quad A + B$

$\diamond \quad \top$ $\qquad$ $\mathsf{abort}(M) \quad \bot$

$$(\lambda x.M)\,N \equiv M[N/x]$$

$$\mathsf{fst}\,\langle M, N\rangle \equiv M \qquad \mathsf{snd}\,\langle M, N\rangle \equiv N$$

$$\mathsf{case}(x.M;\, y.N;\, \mathsf{inl}(O)) \equiv M[O/x]$$

$$\mathsf{case}(x.M;\, y.N;\, \mathsf{inr}(O)) \equiv N[O/y]$$

$$(\lambda x.M)\ N \mapsto M[N/x]$$
$$\mathsf{fst}\ \langle M, N\rangle \mapsto M$$
$$\mathsf{snd}\ \langle M, N\rangle \mapsto N$$
$$\mathsf{case}(x.M;\ y.N;\ \mathsf{inl}(O)) \mapsto M[O/x]$$
$$\mathsf{case}(x.M;\ y.N;\ \mathsf{inr}(O)) \mapsto N[O/y]$$

$$M \xrightarrow[\beta\ \textit{rules}]{} M'$$

$$\lambda x.\mathsf{fst}\langle x,x\rangle \equiv \lambda x.x : A \longrightarrow A$$

$\diamond$ $\quad$ $\mathrm{fst}(x)$

$\mathrm{inl}(\diamond)$ $\quad$ $x(y)(z)$

$\lambda x.x$ $\quad$ x

$\mathrm{inl}(\lambda x.\diamond)$ $\quad$ $\mathrm{case}(\ldots; x)$

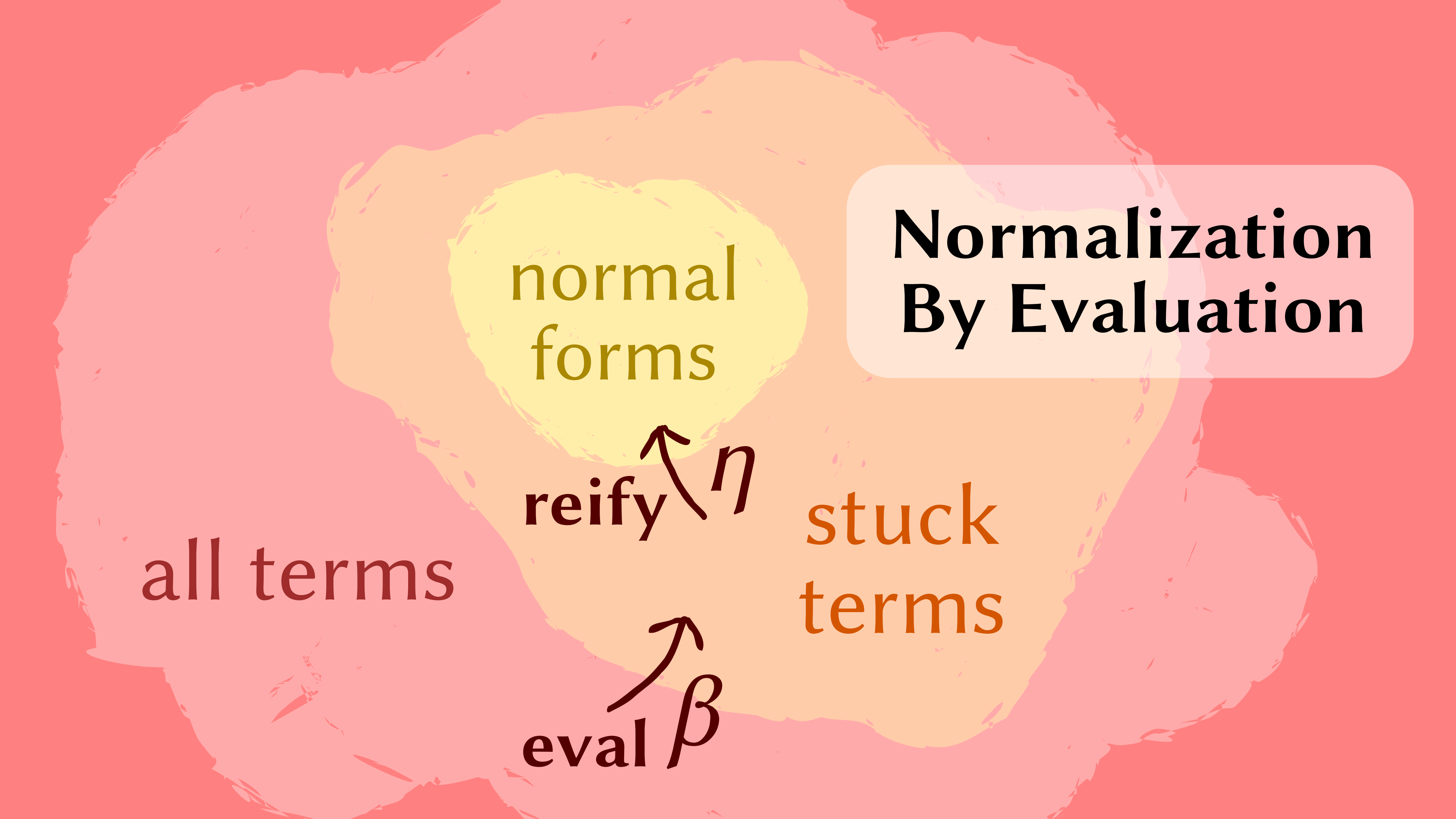
$$F \equiv \lambda x.F(x) : A \longrightarrow B$$

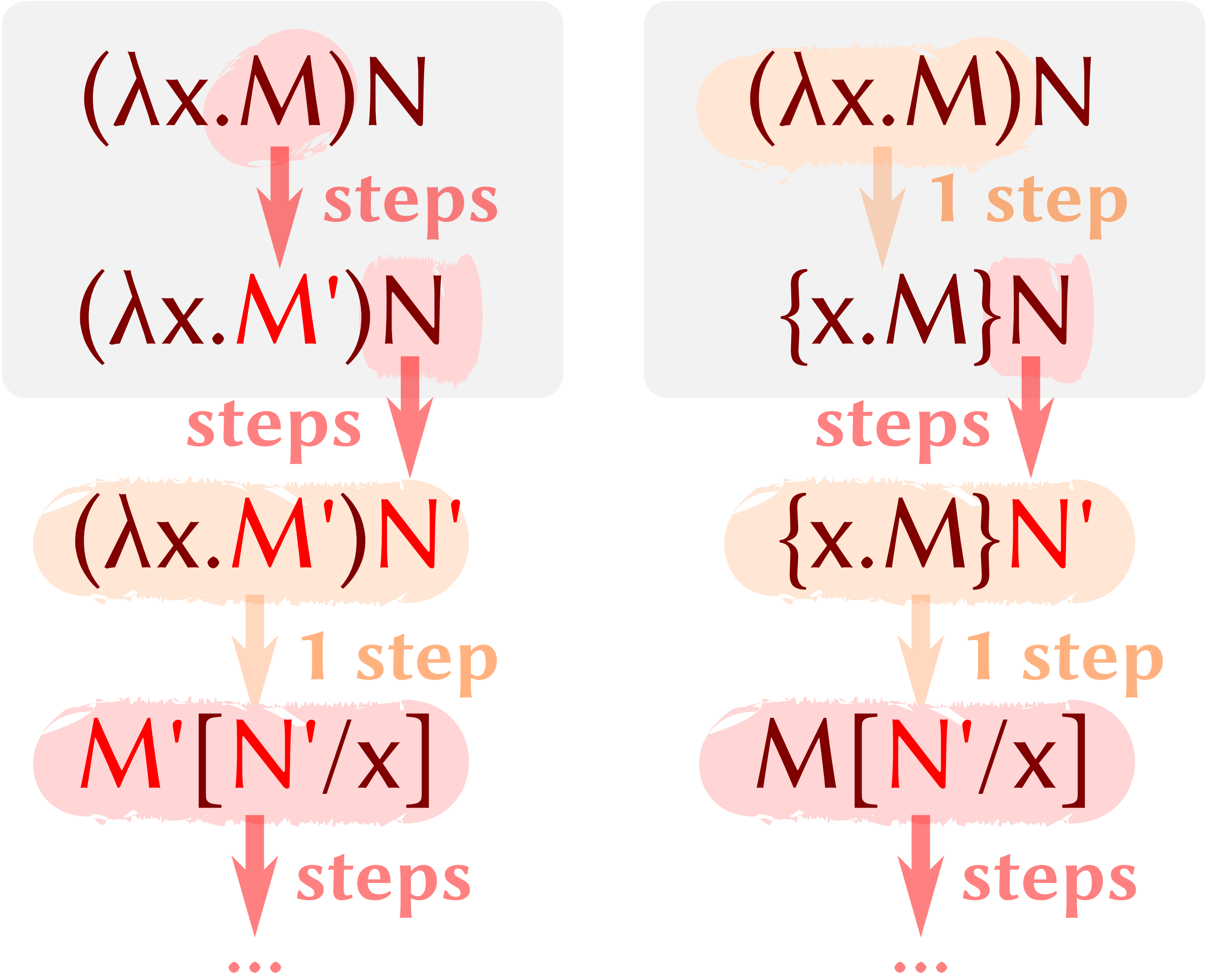
both sides are stuck!

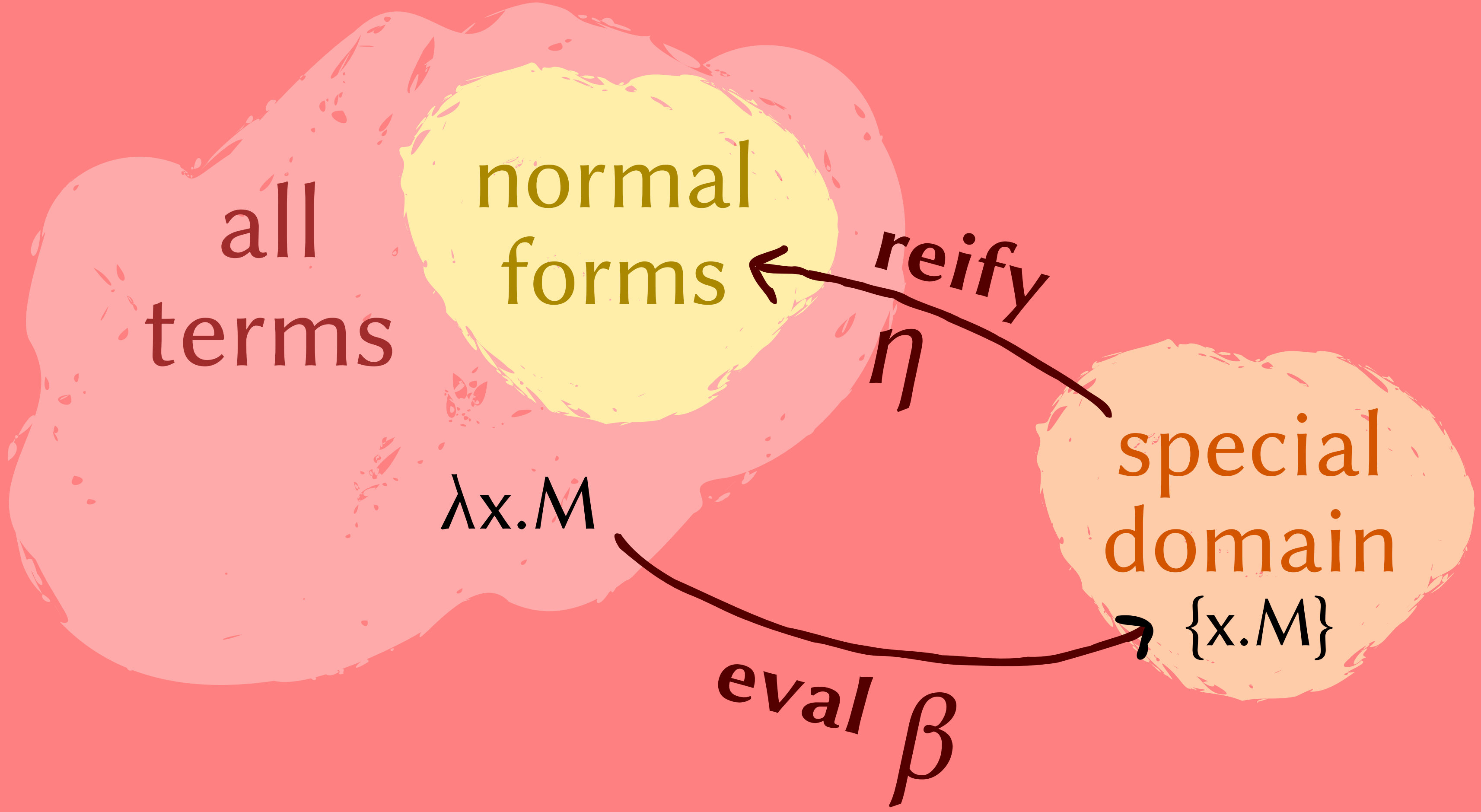
$$P \equiv \langle \mathrm{fst}(P), \mathrm{snd}(P) \rangle : A \times B$$

$$M \equiv \Diamond : \top$$

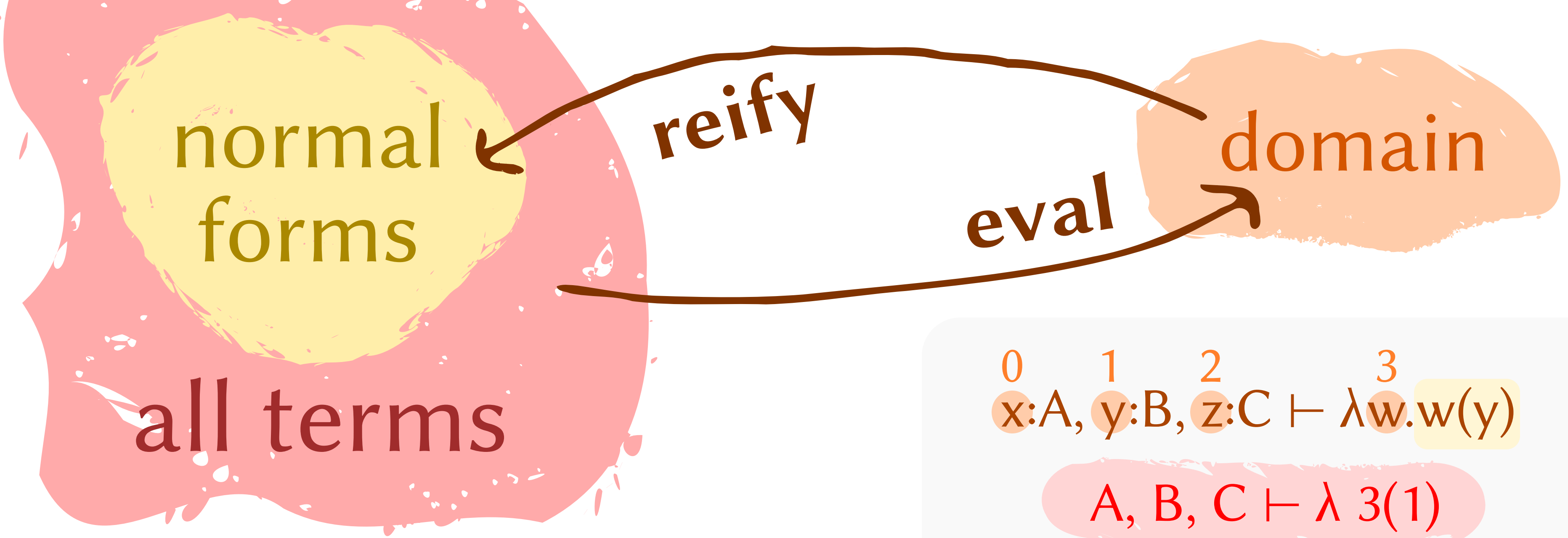
η-expansion


Normalization
By Evaluation
normal
forms
reify η
stuck
terms
all terms
eval β


(λx.M)N
steps
(λx.M')N
steps
(λx.M')N'
1 step
M'[N'/x]
steps
...
(λx.M)N
1 step
{x.M}N
steps
{x.M}N'
1 step
M[N'/x]
steps
...



all terms
normal forms
reify η
special domain
$\lambda x.M$
$\{x.M\}$
eval β



$$\overset{0}{x}{:}A,\ \overset{1}{y}{:}B,\ \overset{2}{z}{:}C \vdash \lambda \overset{3}{w}.w(y)$$

$$A, B, C \vdash \lambda\ 3(1)$$

De Bruijn **levels**: from the **left**

$$x{:}A,\ y{:}B,\ z{:}C \vdash \lambda w.\underset{0}{w}(\overset{2}{y})$$

$$A, B, C \vdash \lambda\ 0(2)$$

De Bruijn **indices**: from the **right**

Trick: use a different scheme in the domain

CUBICAL

$\ldots, p{:}\mathrm{Path}_A(M;N) \vdash p@0 \equiv M : A$

$\ldots, p{:}\mathrm{Path}_A(M;N) \vdash p@1 \equiv N : A$

$\ldots, p{:}\mathrm{Path}_A(M;N) \vdash p@i : A$

$\ldots, p{:}\mathrm{Path}_A(M;N), i{=}0 \vdash p@i \equiv M : A$

$\ldots, p{:}\mathrm{Path}_A(M;N), i{=}1 \vdash p@i \equiv N : A$

$\ldots \vdash \mathrm{loop}_i : S1$

$\ldots, i{=}0 \vdash \mathrm{loop}_i \equiv \mathrm{base} : S1$

$\ldots, i{=}1 \vdash \mathrm{loop}_i \equiv \mathrm{base} : S1$

Normalization by Evaluation

β-reduction + η-expansion

Hereditary Substitution

alternative method

TODO: Cubical Type Theory