Homework 5: Universal Properties

Due 2019/04/17 (Fri) Anywhere on Earth

This homework is to demonstrate the universal properties of the natural number type and propositional truncation. We have seen the universal property that involves $\mathbb{N}$-algebras. As for propositional truncation, we will see a different (but equivalent) universal property. Please send the completed `hw5-handout.agda` to Favonia.

1 Initiality of $\mathbb{N}$

As explained in the lecture, the natural number type is the most general type with a "zero" and a "successor". The new thing is that we are proving this in cubical Agda. (Favonia already defined the path reversal and concatenation for you; check them out!)

Task 1. *Prove the homotopy initiality of $\mathbb{N}$ in cubical Agda.*

Bonus Task 1. *Really complete the proof of initiality, including the bonus part.*

Bonus Task 2. *Show that the initial algebra is unique up to identification. In particular, the underlying type is equivalent to the natural number type. This task requires lots of work. You need to define the composition of two $\mathbb{N}$-homomorphisms, the identity homomorphism, and maybe other things in order to finish the task. You have been warned. (Favonia will still check your partial work.)*

2 Propositional Truncation

Intuitively, the propositional truncation of A (written $\|A\|$) is a new type that remembers only whether A is inhabited and crushes all higher-dimensional structures (up to identification). It is the "best" type at truncation level -1 to approximate A, and we will soon formulate what it means by "best". The type $\|A\|$ can express the concept that there is some element of type A but we do not know what it is. It glues all the elements, if any, into one single cluster. For example, the type $\|\mathbb{N}\|$ (the truncated $\mathbb{N}$) is equivalent to the unit type $\top$ because all the numbers clump together in $\|\mathbb{N}\|$.

Formally, the propositional truncation of type A is defined by the following two constructors:

1. $|-| : A \to \|A\|$
2. $\texttt{prop} : \prod_{x,y:A} \texttt{Path}_{\_.A}(x;y)$

The second constructor forces the propositional truncation itself to be a proposition.

As usual, it is important to understand the universal property of any new definition. We can define the (generalized) algebras for propositional truncations, which will be similar to the $\mathbb{N}$-algebras. However, a more popular universal property is as follows: For any (mere) proposition B and any function $f : A \to B$, there exists a unique function from $\|A\|$ to B such that this triangle commutes: (The uniqueness is up to identification.)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "|-|"'] & B \\ \|A\| \arrow[ur, dashed, "!"'] & \end{tikzcd}$$

In other words, any such function f *factors through* $\|A\|$ and the choice from $\|A\|$ to B is unique.

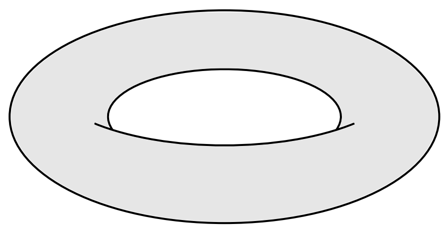
We can formulate this universal property as an equivalence between $A \to B$ and $\|A\| \to B$: For any function from $\|A\|$ to B, we can precompose it with $|-|$ to form a function from A to B. The uniqueness means we can recover the original function from the composed function. This implies there is an equivalence between functions from A to B and those from $\|A\|$ to B, witnessed by the precomposition with $|-|$.

Task 2. *Prove various properties of propositional truncation, including its universal property, in cubical Agda.*

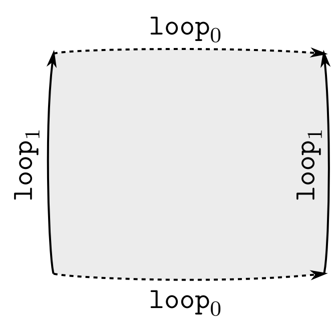
Bonus Task 3. *Prove that $\|A\|$ is equivalent to any proposition satisfying the above universal property. (This one is much simpler than the previous bonus task.)*

Appendix: Syntax of Higher Inductive Types

Extra examples of higher inductive types (inductive types involving path constructors, such as the circle) were added to help you learn the syntax, especially the pattern matching. The theorem in the appendix is to show that the torus is the product of two circles. The torus is the surface of an idealized, smooth donut like this:



The inductive definition is based on the following method to make a torus out of a square. First, grab a piece a paper and mark the edges as follows:



The next step is to glue the two $\mathtt{loop}_0$ to form a tube, bend the tube, and glue the two $\mathtt{loop}_1$ (now two circles) to form a torus. This is how the torus T^2 is defined. Please pay attention to how Favonia used pattern matching on T^2 in the equivalence proof.

Grading

Only one letter grade (without plus or minus) will be assigned to the *entire* homework according to the criterion explained in the syllabus. Bonus tasks are purely for your enjoyment and will not affect your grading (positively or negatively).