Homework 2: Infinity-Groupoids

Due 2019/02/29 (Fri) Anywhere on Earth

The goal of this homework is to learn more about the identification types $\texttt{Id}_A(M; N)$ and formalize some theorems in Agda. You should finish the proofs in `hw2-handout.agda` and email your code as an Agda file to Favonia.

The first theorem is that the concatenation of 2-dimensional loops is always commutative, no matter what the base type is. This implies that all higher homotopy groups are abelian, which is an important result in homotopy theory.

The second theorem is that if the equality between two elements is decidable, then there is at most one identification between any two elements. In other words, the base type is a set. This implies that the natural number type $\mathbb{N}$ is a set because it has decidable equality.

There is no dependency between these two theorems. Feel free to finish them in any order. You can also skip the rest of this PDF and start doing Agda in `hw2-handout.agda` right away! The following are additional hints and explanations for the tasks.

1 Commutative Compositions for Higher Loops

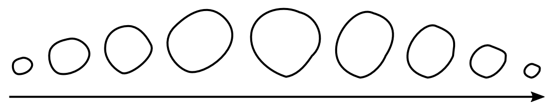
Given a type A with a distinguished element M, the type of (1-dimensional) loops at M in A is

$$\texttt{Id}_A(M; M)$$

which has all the identifications from M to M itself. There is a group structure on $\texttt{Id}_A(M; M)$ that is similar to the groupoid structure we constructed in class. We can iterate this process to construct higher-dimensional loop spaces, and their group structures are always *abelian.* We will focus on the second dimension in this task. To start with, a 2-*dimensional loops* is a loop at $\texttt{refl}(M)$, whose type is

$$\Omega^2(A; M) :\equiv \texttt{Id}_{\texttt{Id}_A(M;M)}(\texttt{refl}(M); \texttt{refl}(M)).$$

Imagine an extremely elastic jumping rope. A 2-dimensional loop is the trace of this jumping rope moving from a rest position (reflexivity) and back to the rest position (reflexivity).

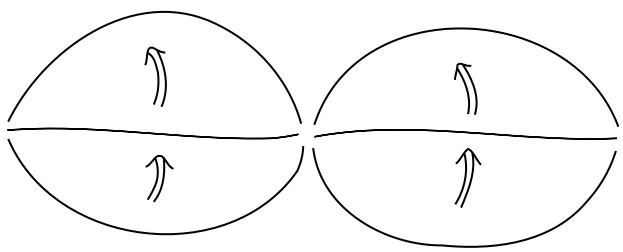


One important result is that the concatenation of 2-dimensional loops is commutative (up to identification):

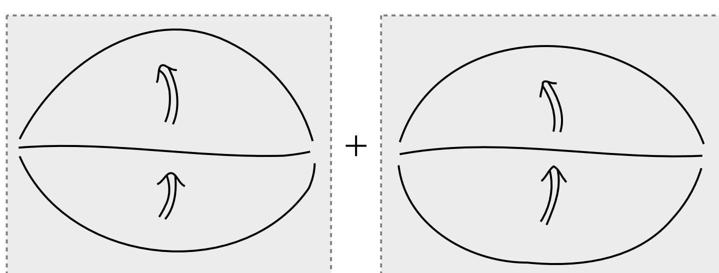
Theorem 1 (a special case of Eckmann–Hilton)**.**

$$\prod_{x:A} \prod_{(p:\Omega^2(A;x))} \prod_{(q:\Omega^2(A;x))} \mathtt{Id}_{\Omega^2(A;x)}(p \cdot q; q \cdot p).$$

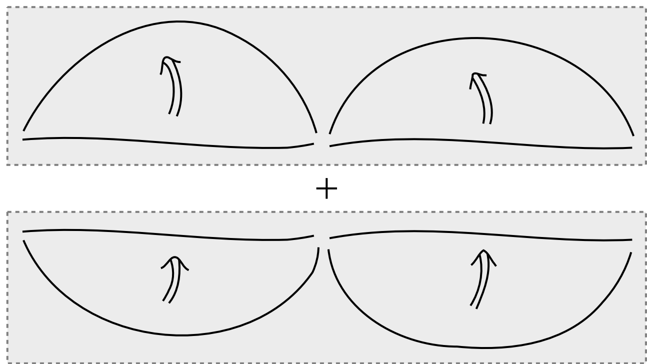
Here is one pictorial proof: Consider four 2-dimensional paths (traces of jumping ropes) aligned as follows:



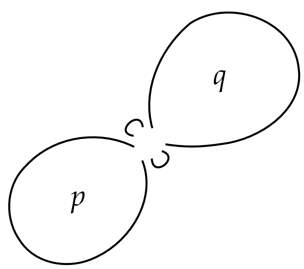
There are two canonical ways to compose these 2-dimensional paths together. You can vertically compose two paths and then horizontally compose two resulting paths:



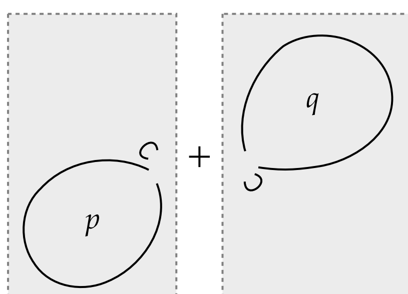
On the other hand, you can compose these paths row-wise first:



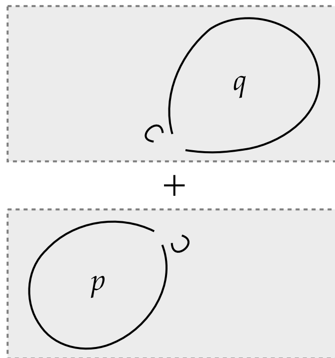
The idea is that the eventual path should not depend on how you compose individual parts. Now, think about the special case where the bottom left is $p : \Omega^2(A; x)$, the top right is $q : \Omega^2(A; x)$, and the remaining two are the reflexivity $\texttt{refl}(\texttt{refl}(x))$.



The first method gives you $p \cdot q$:



The second gives you $q \cdot p$ (by reading from the top to the bottom):



Because the two methods should give the same result, there is an identification between $p \cdot q$ and $q \cdot p$. We are done! Well, at least intuitively.

In classical homotopy theory, one can prove this by the "floating islands" argument [1, the picture on p. 340] or the *Eckmann–Hilton argument*, a general principle in algebra. It turns out we additionally have a relatively short proof in our type theory and Agda. The pictorial proofs, the algebraic magic, and the type-theoretic code are just different avatars of the same concept. The connection among these seemingly different proofs was claimed to be inspiring by some experts.

The major difficulty in type theory is that we cannot simply do pattern matching on $p : \Omega^2(A; x)$ or $q : \Omega^2(A; x)$ without the forbidden axiom K (uniqueness of identification proofs). The motive in $\mathtt{J}[x.y.p]$ requires the endpoints x and y to be "free", but here, the end points of p or q are locked up. The banned axiom K would enable you to handle the cases where the end points are locked, at the price of killing interesting higher-dimensional structures. We love higher-dimensional structures.

Task 1. *Finish the subtasks in the Agda file. The subtasks lead to a possible proof, and you are free to replace them with something else. The important thing is to prove the final theorem* `eckmann-hilton` *in Agda.* **(Hint)** *Read the proof of [2, Theorem 2.1.6].*

2 Decidability Implies Setness

The next theorem is to explore another interesting fact about identifications. If we have a procedure to decide whether two elements are equal, then that type is a set. To start with, the type of such a procedure d for some type A is:

$$\prod_{x:A} \prod_{y:A} \mathtt{Id}_A(x; y) + (\neg \mathtt{Id}_A(x; y))$$

This means the procedure will either gives a proof of $\mathtt{Id}_A(x; y)$ of a refutation of it for any pair $x, y : A$. We can define setness as follows:

$$\mathtt{is\text{-}set}(A) :\equiv \prod_{x:A} \prod_{y:A} \prod_{p:\mathtt{Id}_A(x;y)} \prod_{q:\mathtt{Id}_A(x;y)} \mathtt{Id}_{\mathtt{Id}_A(x;y)}(p; q)$$

which says that any two paths between any two elements can be identified. (The definition of `is-set` is different from (but equivalent to) what we covered on 2020/02/13. It is chosen to make your life easier.) We then have the following theorem for any type A:

Theorem 2 (Hedberg)**.**

$$\left(\prod_{x:A} \prod_{y:A} \mathtt{Id}_A(x; y) + (\neg \mathtt{Id}_A(x; y)) \right) \to \mathtt{is\text{-}set}(A).$$

The idea is to construct a function $s : \mathtt{Id}_A(x; y) \to \mathtt{Id}_A(x; y)$ for any $x, y : A$ such that

- It is homotopic to the identity function. That is, you can inhabit this type for any elements $x, y : A$:

$$\prod_{p:\mathtt{Id}_A(x;y)} \mathtt{Id}_{\mathtt{Id}_A(x;y)}(s(p); p)$$

- It is (weakly) constant. That is, you can inhabit this type for any elements $x, y : A$:

$$\prod_{p:\mathtt{Id}_A(x;y)} \prod_{q:\mathtt{Id}_A(x;y)} \mathtt{Id}_{\mathtt{Id}_A(x;y)}(s(p); s(q))$$

It then follows that any two paths p and q can be identified. How can we construct such a magical function s? We will exploit the fact that the procedure d deciding equality is able to pick an identification *continuously and functorially* for any identified elements. To begin with, let's define a function $r : \mathtt{Id}_A(x; y) \to \mathtt{Id}_A(x; y)$:

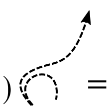
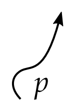
$$r(p) :\equiv \mathtt{case}(q.q; \bar{p}.\mathtt{abort}(\bar{p}\ p); d\ x\ y)$$

The idea is that, if we already know x and y are identified, then we can invoke the procedure d to obtain an identification between them. This may seem to be a total waste of time, because we already had an identification, but the magic of such a detour is that the identification chosen by the procedure d *does not* depend on the input identification p! Thus, r is weakly constant by construction and it suffices to show r is homotopic to the identity function.

While it is true that r is homotopic to the identity, we will use the following function s instead of r to simplify the calculation:

$$s(p) :\equiv r(\texttt{refl}(x))^{-1} \cdot r(p)$$

The function s is also weakly constant because r is weakly constant. For the other condition, here is a pictorial proof showing that s is homotopic to the identity function: The equation can be visualized as follows, where the path $s(p)$ on the left hand side is the concatenation of the $r(\texttt{refl})^{-1}$ and $r(p)$:

$s(p)$  $=?$  p

Why does this equation hold? It heavily depends on the fact that s is continuous. We can continuously shrink p to `refl`, and the equation will become:

$s(\texttt{refl})$  $=$  `refl`

Two sides agree because $r(\texttt{refl})^{-1}$ and $r(\texttt{refl})$ cancel each other.

Task 2. *Finish the subtasks in the Agda file. Again, you are free to replace intermediate lemmas, as long as you can prove the final theorem* **`hedberg`** *in Agda.* **(Hint)** *The above proof is already one of the most polished. I do not think the Internet can help you much, except the Agda documentation.*

3 Grading

Only one letter grade (without plus or minus) will be assigned to the *entire* homework according to the criterion explained in the syllabus.