

# Re: Homework 1: J

Due 2019/05/01 (Fri) Anywhere on Earth

Please implement **one of** the following functions using J (in the pre-cubical era, without any cubical features) and show that it is well-typed:

1. Path concatenation of the following type:

$$\prod_{A:\mathcal{U}} \prod_{x:A} \prod_{y:A} \prod_{z:A} \text{Id}_A(x; y) \rightarrow \text{Id}_A(y; z) \rightarrow \text{Id}_A(x; z)$$

2. Congruence/functoriality of the following type:

$$\prod_{A:\mathcal{U}} \prod_{B:\mathcal{U}} \prod_{f:A \rightarrow B} \prod_{x:A} \prod_{y:A} \text{Id}_A(x; y) \rightarrow \text{Id}_B(f(x); f(y))$$

You should write out your term and then the entire derivation. See the next page for an example. Please be attentive to the following two things:

1. The difference between variables inside a type theory (“ $x:A$ ” in the hypothetical judgment “ $x:A \vdash M : B$ ”) and metavariables outside the theory ( $\mathcal{D}$  and  $\mathcal{J}$  in “let  $\mathcal{D}$  be a derivation of some judgment  $\mathcal{J} \dots$ ”).
2. How the contexts change between judgments in a rule.

In lectures, we intentionally removed some premises in certain rules, such as “ $A : \mathcal{U}$ ” from the Id-intro rule, because the removed premise is implied by other premises. In other words, the removal does not materially change the type theory. You can choose any (correct) version in this homework. However, I highly recommend using the more verbose version when there is a choice, which is demonstrated in the next page.

**The type:**

$$\prod_{A:\mathcal{U}} \prod_{x:A} \prod_{y:A} \text{Id}_A(x, y) \rightarrow \text{Id}_A(y, x)$$

**The term:**

$$\lambda(A:\mathcal{U}).\lambda(x:A).\lambda(y:A).\lambda(p:\text{Id}_A(x,y)).\text{J}[x'.y'.p'.\text{Id}_A(y';x')](x'.\text{refl}(x');p)$$

**The derivation:** Let

$$\begin{aligned}\Gamma &= A:\mathcal{U}, x:A, y:A, p:\mathrm{Id}_A(x, y) \\ \Gamma' &= \Gamma, x':A, y':A, p':\mathrm{Id}_A(x', y')\end{aligned}$$

and

$$\mathcal{D} = \frac{\frac{\Gamma' \vdash A : \mathcal{U}}{\text{var}} \quad \frac{\Gamma' \vdash y' : A}{\text{var}} \quad \frac{\text{var} \quad \text{var}}{\Gamma' \vdash x' : A}}{\Gamma' \vdash \text{Id}_A(y', x') : \mathcal{U}} \quad \text{Id-form}$$

Note that  $\Gamma$ ,  $\Gamma'$ , and  $\mathcal{D}$  are metavariables. Here is the complete derivation tree:

$$\begin{array}{c}
\frac{\mathcal{D} \quad \frac{\Gamma, x':A \vdash A : \mathcal{U} \quad \frac{\Gamma, x':A \vdash x' : A}{\text{Id-intro}} \quad \frac{\Gamma \vdash x : A}{\text{var}} \quad \frac{\Gamma \vdash y : A}{\text{var}} \quad \frac{\Gamma \vdash p : \text{Id}_A(x; y)}{\text{Id-elim}}}{\Gamma \vdash A : \mathcal{U}} \\
\hline
\Gamma' \vdash \text{Id}_A(y'; x') : \mathcal{U} \\
\hline
\Gamma \vdash \text{J}[x', y'.p'.\text{Id}_A(y'; x')](x'.\text{refl}(x'); p) : \text{Id}_A(y; x) \quad \rightarrow\text{-intro} \\
\hline
A:\mathcal{U}, x:A, y:A \vdash \lambda(p:\text{Id}_A(x; y)).\text{J}[x'.y'.p'.\text{Id}_A(y'; x')](x'.\text{refl}(x'); p) : \text{Id}_A(y; x) \quad \rightarrow\text{-intro} \\
\hline
A:\mathcal{U}, x:A \vdash \lambda(y:A).\lambda(p:\text{Id}_A(x; y)).\text{J}[x'.y'.p'.\text{Id}_A(y'; x')](x'.\text{refl}(x'); p) : \prod_{y:A} \text{Id}_A(x; y) \rightarrow \text{Id}_A(y; x) \quad \rightarrow\text{-intro} \\
\hline
A:\mathcal{U} \vdash \lambda(x:A).\lambda(y:A).\lambda(p:\text{Id}_A(x; y)).\text{J}[x'.y'.p'.\text{Id}_A(y'; x')](x'.\text{refl}(x'); p) : \prod_{x:A} \prod_{y:A} \text{Id}_A(x; y) \rightarrow \text{Id}_A(y; x) \quad \rightarrow\text{-intro} \\
\hline
\vdash \lambda(A:\mathcal{U}).\lambda(x:A).\lambda(y:A).\lambda(p:\text{Id}_A(x; y)).\text{J}[x'.y'.p'.\text{Id}_A(y'; x')](x'.\text{refl}(x'); p) : \prod_{A:\mathcal{U}} \prod_{x:A} \prod_{y:A} \text{Id}_A(x; y) \rightarrow \text{Id}_A(y; x)
\end{array}$$